# Emergence of large scale structures in barotropic turbulence

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# Turbulent flows are organized into vortices and jets

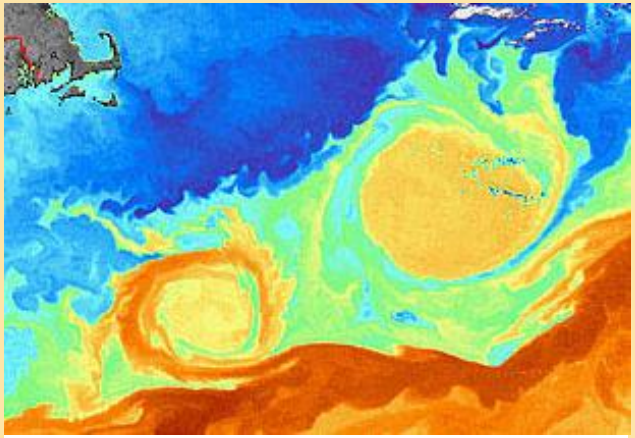
banded jets

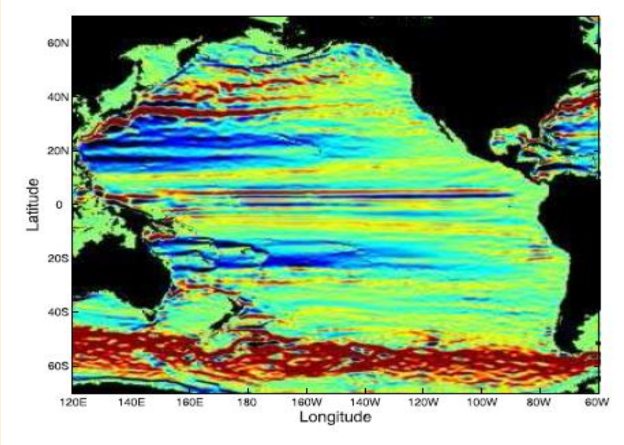
NASA

Great Red Spot

NASA

ocean jets & rings

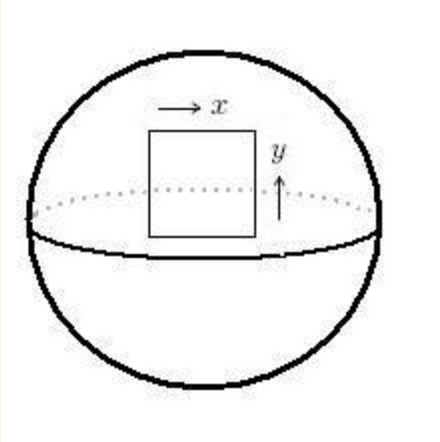





(Richards et al 2006)

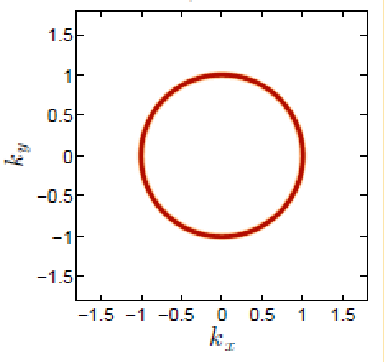
# Simplest model: barotropic flow on a beta-plane

- Simplest setting — non-divergent flow in a doubly periodic β-plane channel




$$\zeta = \partial_x v - \partial_y u = \Delta\psi \qquad \left(\partial_t + \vec{u}\cdot\nabla\right)\zeta + \beta v = -r\zeta - \nu\nabla^4\zeta + \xi(t)$$

- Spatially homogeneous forcing that is delta-correlated in time




$$\left\langle \xi(x_1,y_1,t_1)\xi(x_2,y_2,t_2)\right\rangle = \delta(t_1-t_2)\Xi(x_1,x_2,y_1,y_2)$$

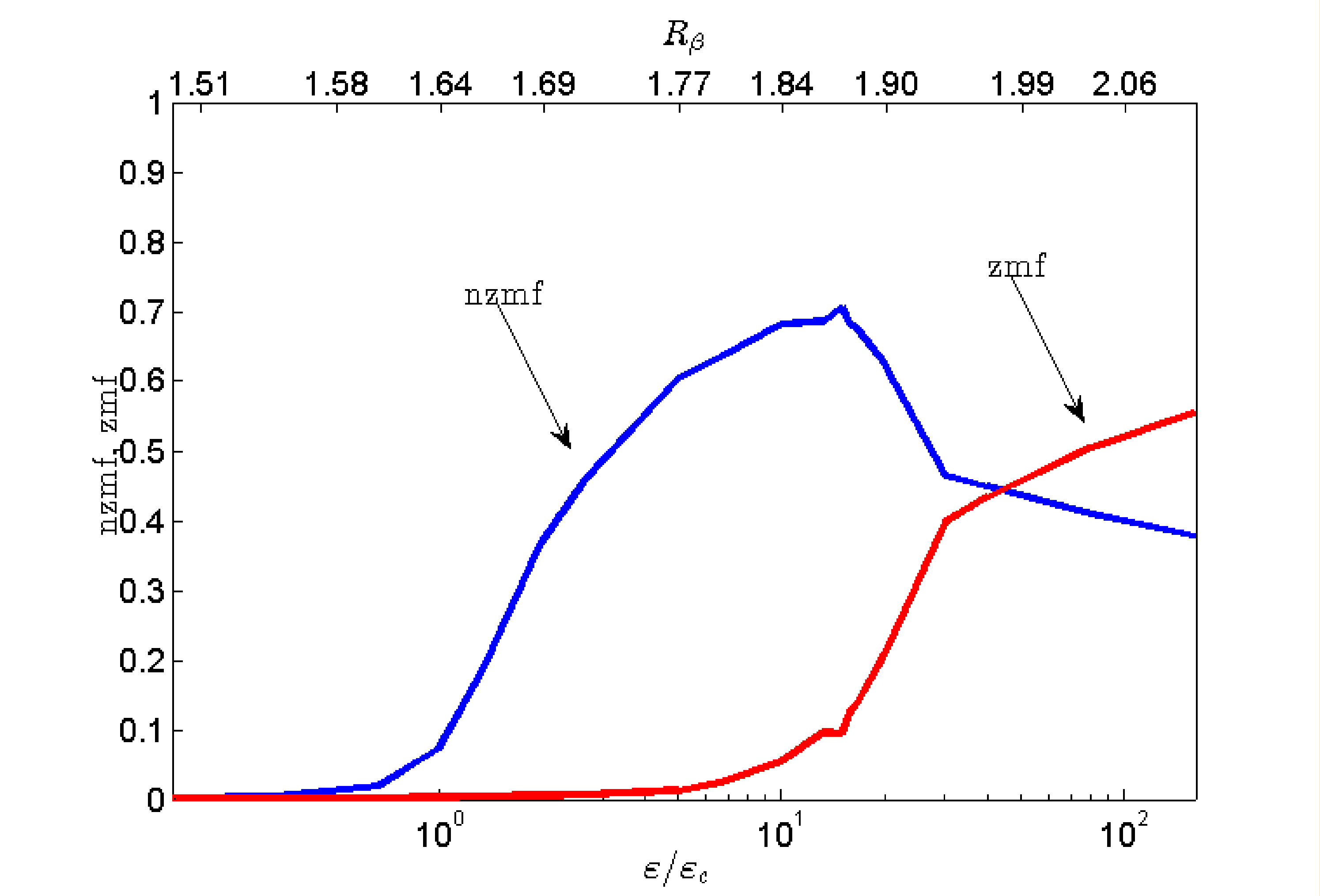
$$\Xi(x_1,x_2,y_1,y_2) = \sum \hat{\Xi}(k,l)e^{ik(x_1-x_2)+il(y_1-y_2)}$$

- Isotropic forcing injecting energy at rate $\varepsilon$ in a narrow ring at $K_f$

$$K_f = 10\ ,\ \Delta K_f = 1\ ,\ \beta = 10\ ,\ r = 0.01\ ,\ \nu = 10^{-6}$$

# Two regime transitions in the flow

$$\text{zmf} = \frac{\sum_{l:l<K_f} \hat{E}(k=0,l)}{\sum \hat{E}(k,l)} \qquad \text{nzmf} = \frac{\sum_{k,l:K<K_f} \hat{E}(k,l)}{\sum \hat{E}(k,l)} - \text{zmf}$$



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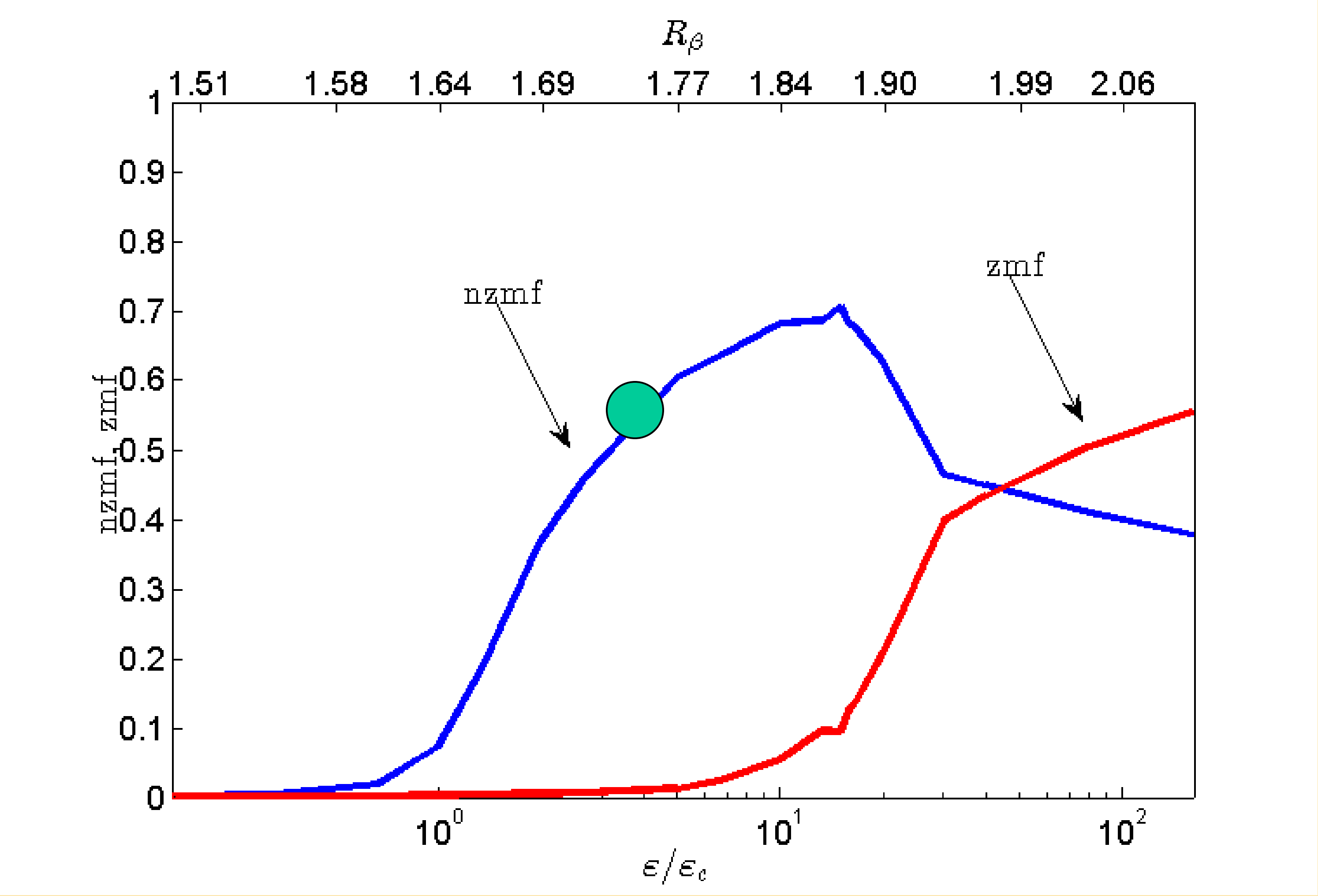


# Non-zonal westward propagating coherent structures

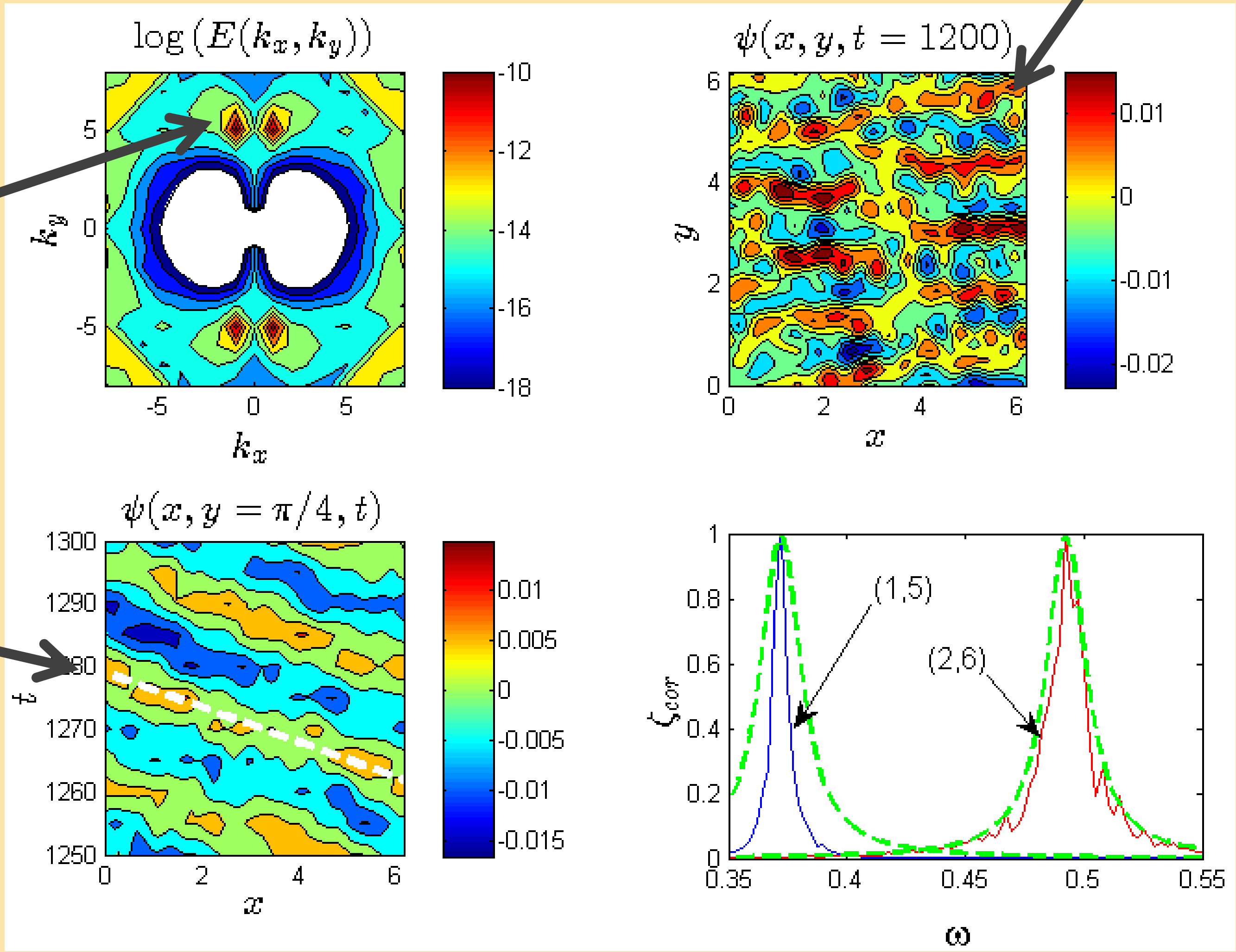
$\varepsilon/\varepsilon_c = 4$

$R_\beta = 1.62$

peak at
$(k_x, k_y) = (1, 5)$

a $(k_x, k_y) = (1, 5)$ structure

Rossby wave phase speed



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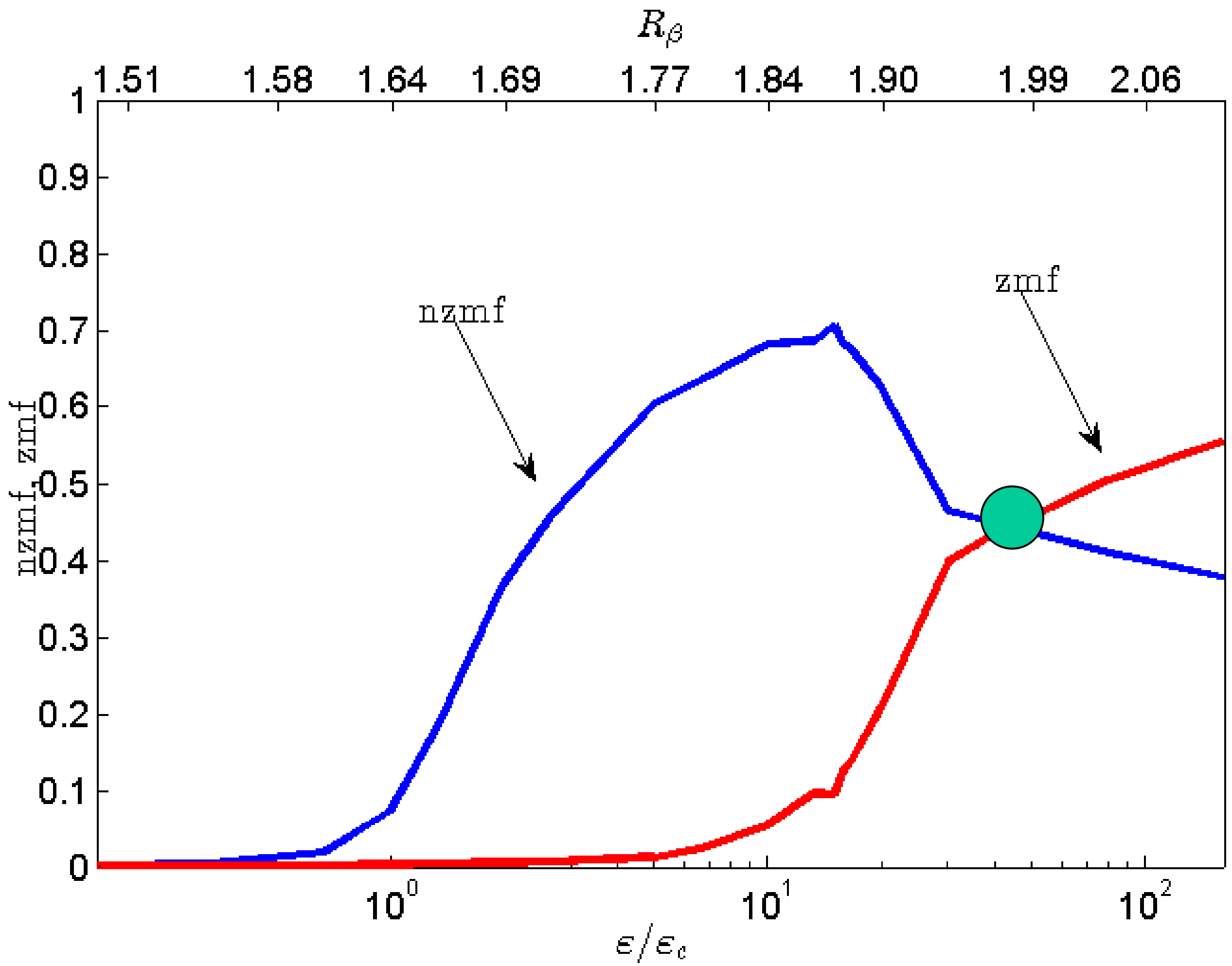
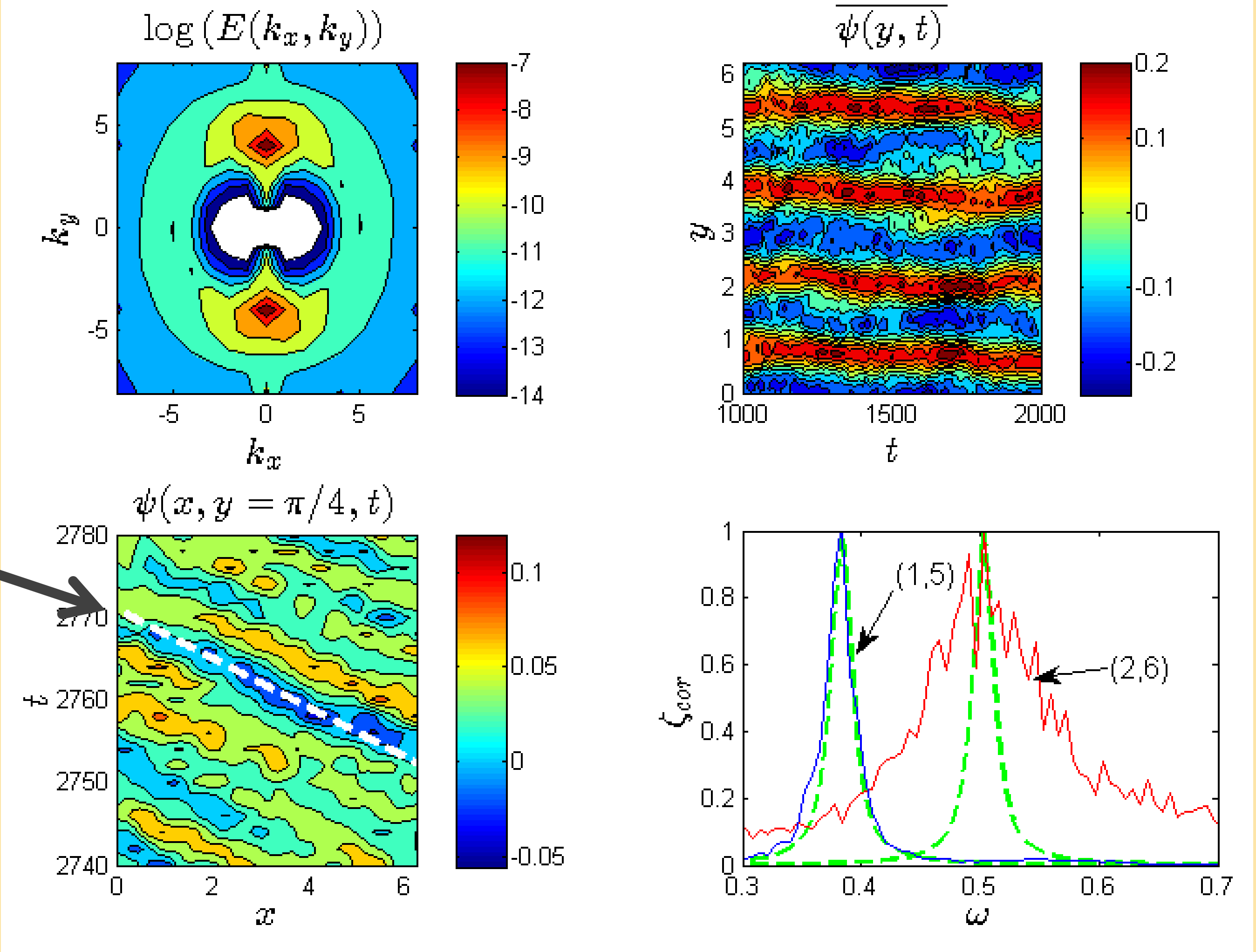


# Zonal jets emerge, NZCS persist but slow down

$\varepsilon/\varepsilon_c = 50$

$R_\beta = 1.84$

slower than Rossby wave



# Our goal

Develop a theory that accurately predicts:

- The regime transitions in the flow (NZCS, jet emergence)

- The characteristics (scale, amplitude, phase speed) of the emergent structures

- Provides an explanation for the dynamics underlying structure formation

# Theory for the statistical state dynamics: S3T

- Stochastic Structural Stability Theory (S3T)

  Farrell & Ioannou 2003

  Related: Cumulant Expansion (CE2)  Marston et al. 2008

- Variables : mean + deviation :  $\zeta = Z + \zeta'$

$$\left(\partial_t + U\partial_x + V\partial_y\right)\zeta' + \left(\beta + Z_y\right)v' + Z_x u' \;=\; \xi - r\zeta' - \nu\nabla^4\zeta' + \left\langle \vec{u}'\cdot\nabla\zeta'\right\rangle - \vec{u}'\cdot\nabla\zeta'$$

$$\left(\partial_t + U\partial_x + V\partial_y\right)Z + \beta V \;=\; -\partial_x\left\langle u'\zeta'\right\rangle - \partial_y\left\langle v'\zeta'\right\rangle - rZ - \nu\nabla^4 Z$$

# Cumulant expansion

$$\left(\partial_t + U\partial_x + V\partial_y\right)Z + \beta V = -\partial_x\left\langle u'\zeta'\right\rangle - \partial_y\left\langle v'\zeta'\right\rangle - rZ - \nu\nabla^4 Z$$

evolution of 1st cumulant $Z = \left\langle \zeta \right\rangle$

$$-\partial_x\left\langle u'\zeta'\right\rangle - \partial_y\left\langle v'\zeta'\right\rangle = F(C)$$

$$C = \left\langle \zeta_1' \zeta_2' \right\rangle \; , \; \zeta_i' = \zeta'(x_i, y_i, t)$$

$$\frac{dZ}{dt} = -\left(U\partial_x + V\partial_y\right)Z - \beta V - rZ - \nu\nabla^4 Z + F(C)$$

$$\partial_t \zeta_i' = A_i \zeta_i' + \xi_i + \left\langle \vec{u}_i' \cdot \nabla \zeta_i' \right\rangle - \vec{u}_i' \cdot \nabla \zeta_i'$$

$$A_i = -\left(U_i \partial_{x_i} + V_i \partial_{y_i}\right) - \left(\beta + Z_{y_i}\right)\partial_{x_i} \nabla_i^{-2} + Z_{x_i} \partial_{x_i} \nabla_i^{-2} - r - \nu\nabla_i^4$$

evolution of 2nd cumulant $C = \left\langle \zeta_1' \zeta_2' \right\rangle$

$$\frac{dC}{dt} = \left(A_1 + A_2\right)C + \Xi + G\left(\left\langle \zeta_1' \zeta_2' \zeta_3' \right\rangle\right)$$

# Second order closure of cumulant expansion

CE2: $G\left(\left\langle \zeta_1'\zeta_2'\zeta_3' \right\rangle\right) = 0 \leftrightarrow \left\langle \vec{u}_i' \cdot \nabla \zeta_i' \right\rangle - \vec{u}_i' \cdot \nabla \zeta' = 0$ ignore the eddy-eddy

S3T: $G\left(\left\langle \zeta_1'\zeta_2'\zeta_3' \right\rangle\right) = \Gamma - r_{eff} C \leftrightarrow \left\langle \vec{u}_i' \cdot \nabla \zeta_i' \right\rangle - \vec{u}_i' \cdot \nabla \zeta' = \gamma(t) - r_{eff} \zeta'$

eddy-eddy term acting as stochastic forcing and dissipation

- Ergodic assumption: ensemble average = time average

$$\boxed{\frac{dZ}{dt} = -\left(U\partial_x + V\partial_y\right)Z - \beta V - rZ - \nu\nabla^4 Z + F(C)} \quad \text{———} \quad \boxed{\frac{dC}{dt} = \left(A_1 + A_2\right)C + \Xi}$$

Closed, deterministic system for the coherent flow & eddy statistics

# Regime transitions in turbulence as an instability

$$\left.\begin{aligned}\frac{dC}{dt} &= (A_1 + A_2)C + \Xi = 0 \\ \frac{dZ}{dt} &= -(U\partial_x + V\partial_y)Z - \beta V - rZ + F(C) = 0\end{aligned}\right\} \to Z^E, C^E$$

- Linearization: study the evolution of small perturbations in the mean structure $\delta Z$ and in the eddy statistics $\delta C$ :

$$Z^E = 0, C^E = \Xi / 2r \qquad \frac{d}{dt}\begin{pmatrix}\delta Z \\ \delta C\end{pmatrix} = L(Z^E, C^E)\begin{pmatrix}\delta Z \\ \delta C\end{pmatrix}$$

turbulent eddies

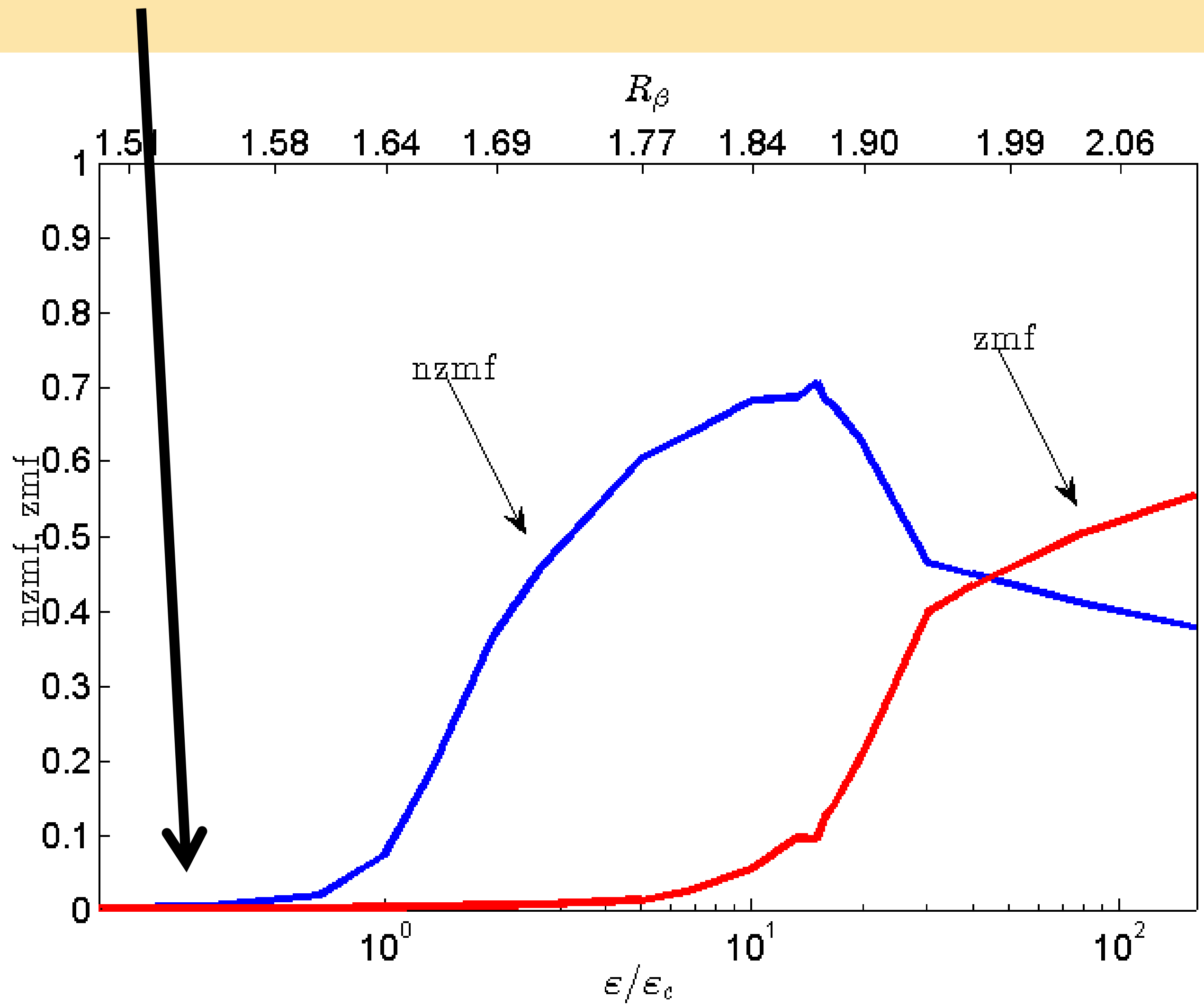
intensify the mean structure through upgradient vorticity fluxes

organizes the eddies so that the eddy fluxes are reinforced

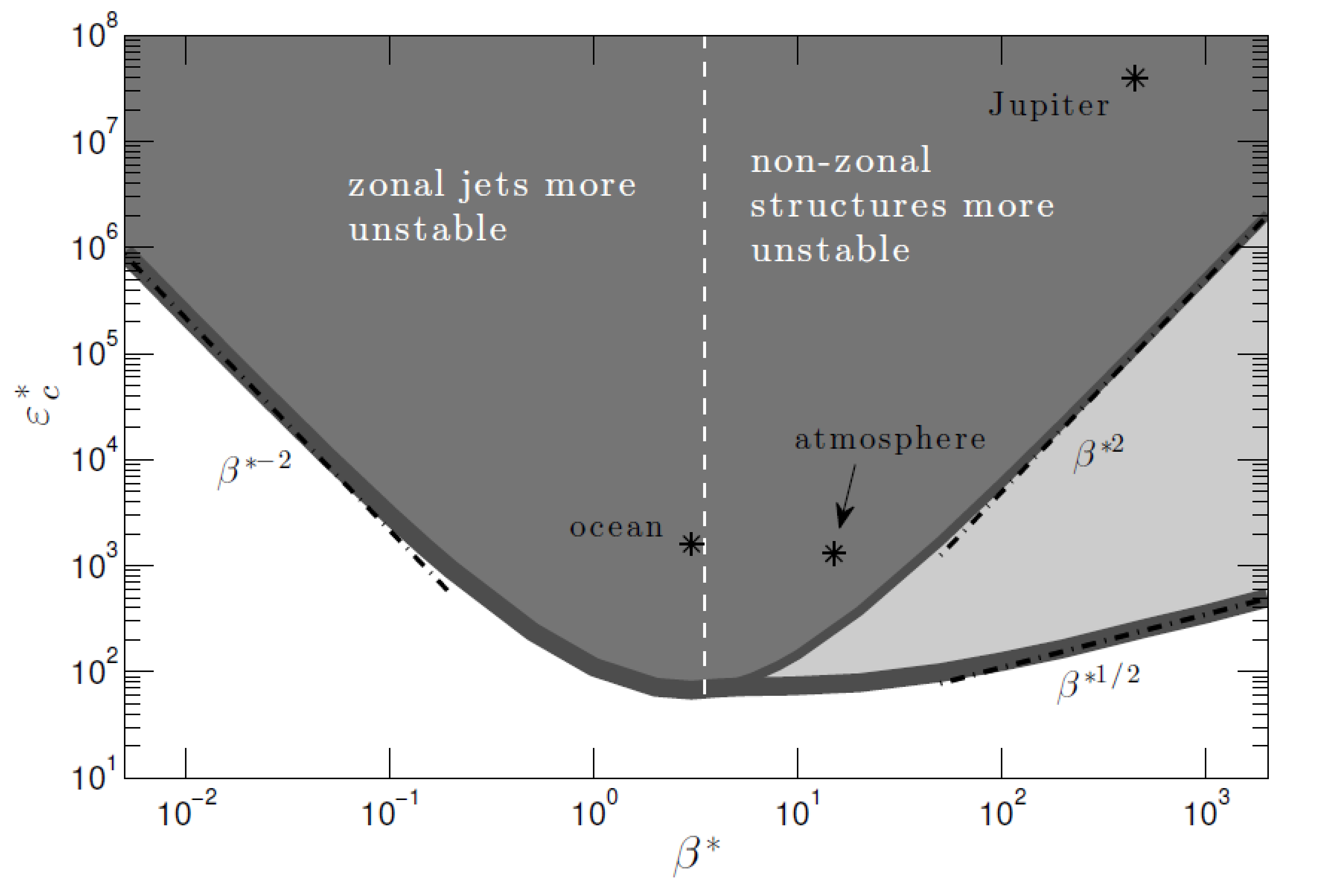
mean structure

# Stability of homogeneous equilibrium

$Z^E = 0, C^E = \Xi / 2r$

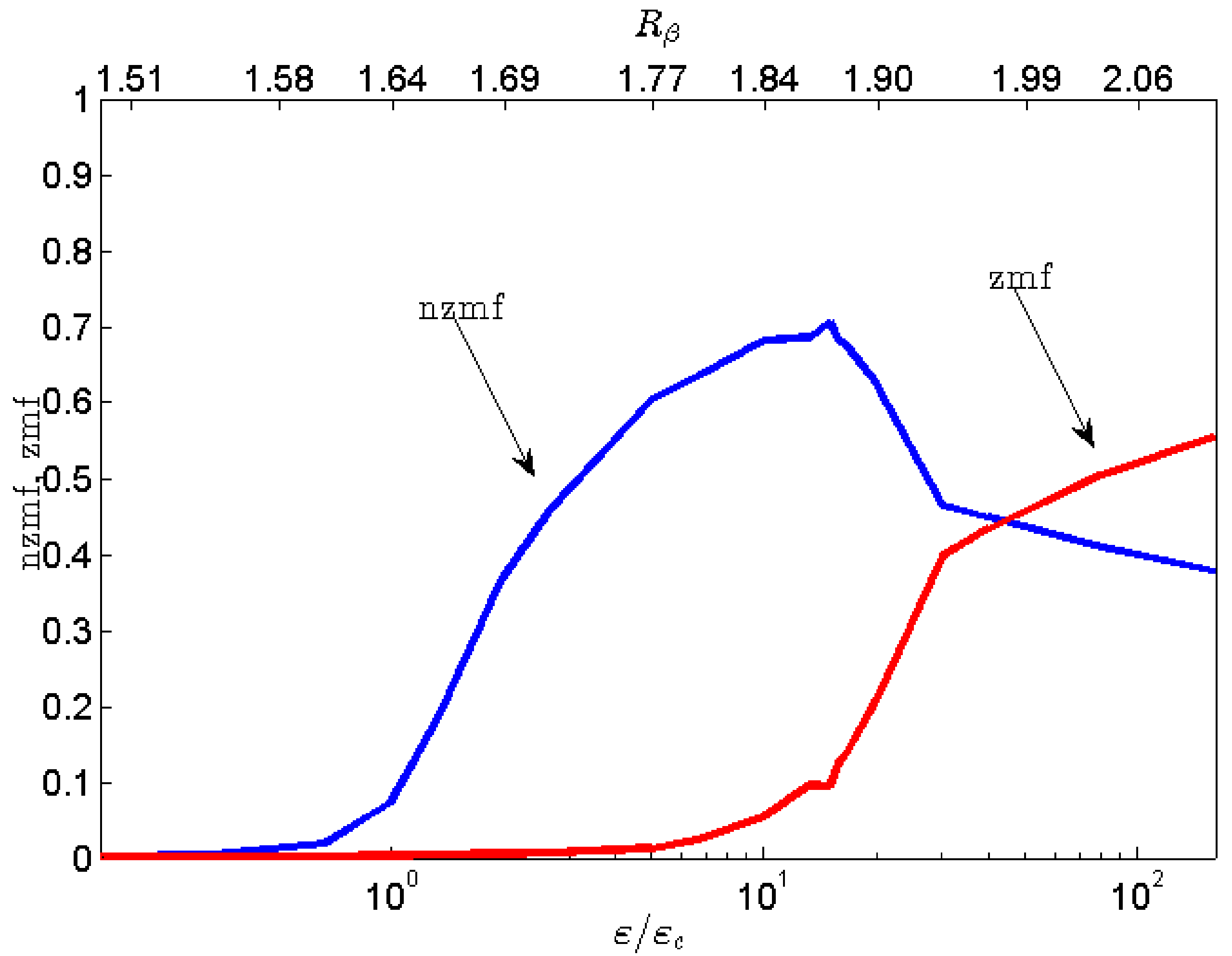


# Critical curve

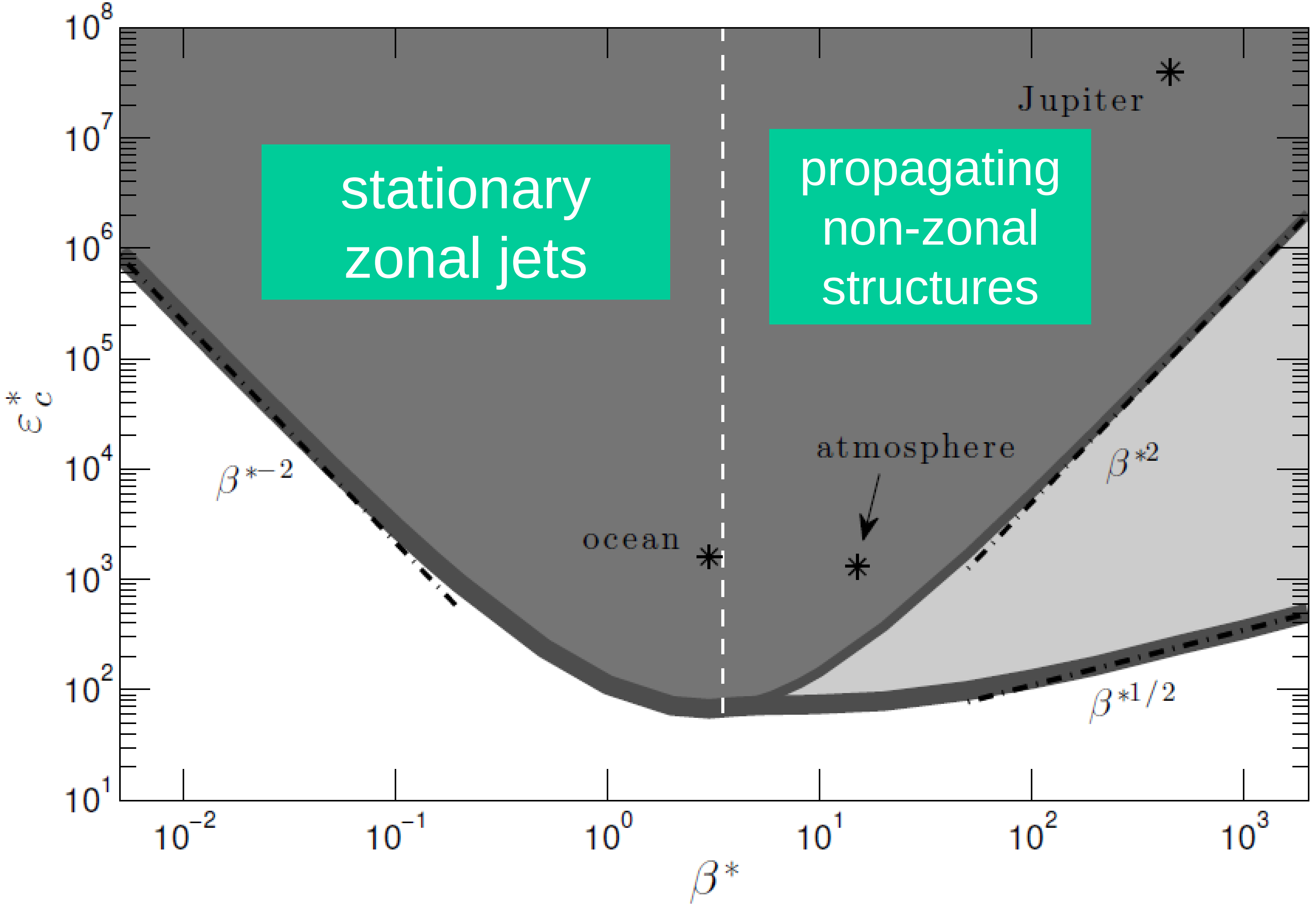


# S3T predicts the first regime transition

$$\text{zmf} = \frac{\sum_{l:l<K_f} \hat{E}(k=0,l)}{\sum \hat{E}(k,l)} \qquad \text{nzmf} = \frac{\sum_{k,l:K<K_f} \hat{E}(k,l)}{\sum \hat{E}(k,l)} - \text{zmf}$$



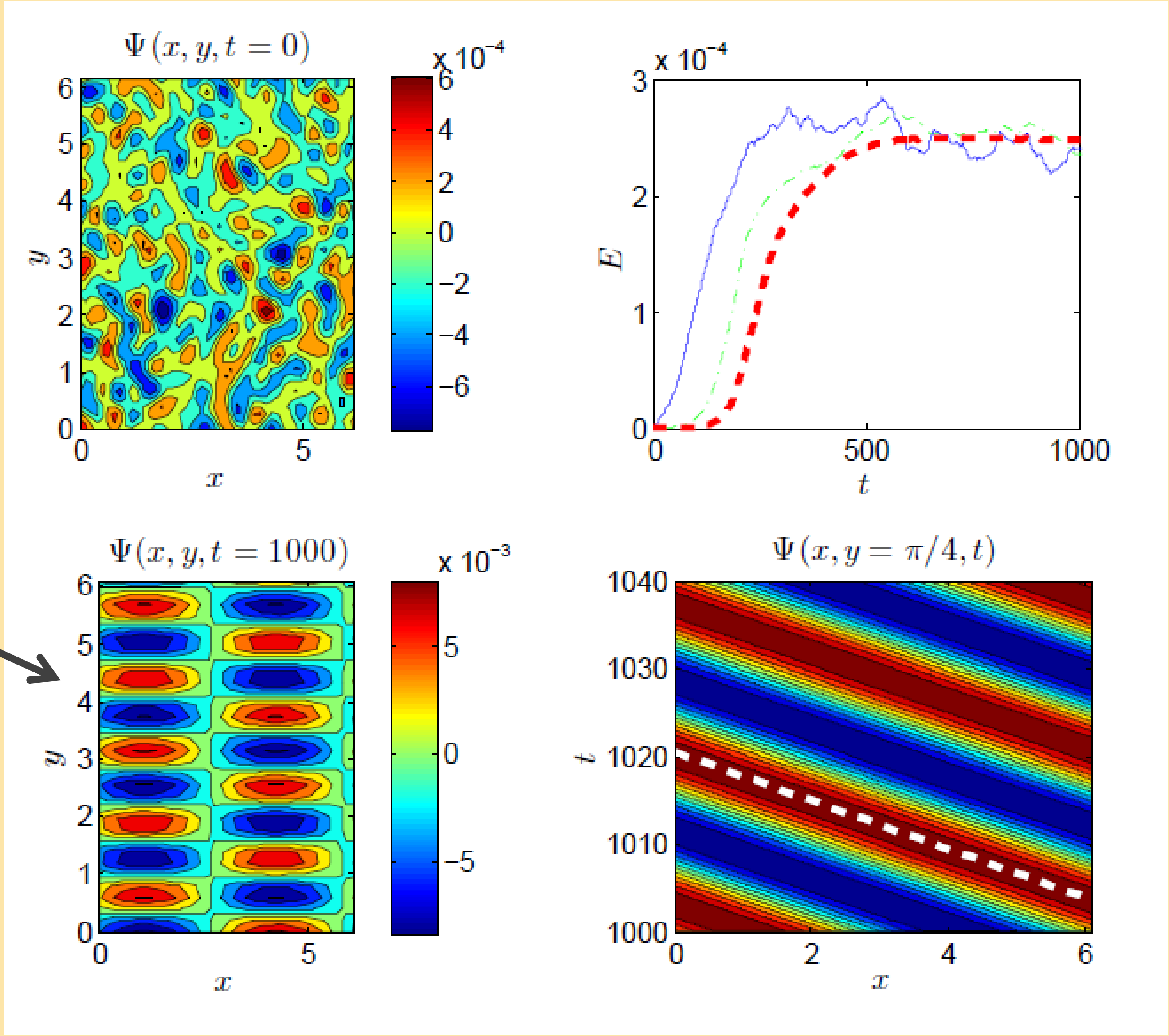
# Most unstable structures



# Equilibration of instabilities: emergent structures

$\tilde{\beta} = 100$
$\tilde{\varepsilon}/\tilde{\varepsilon}_c = 4$
$R_\beta = 1.62$

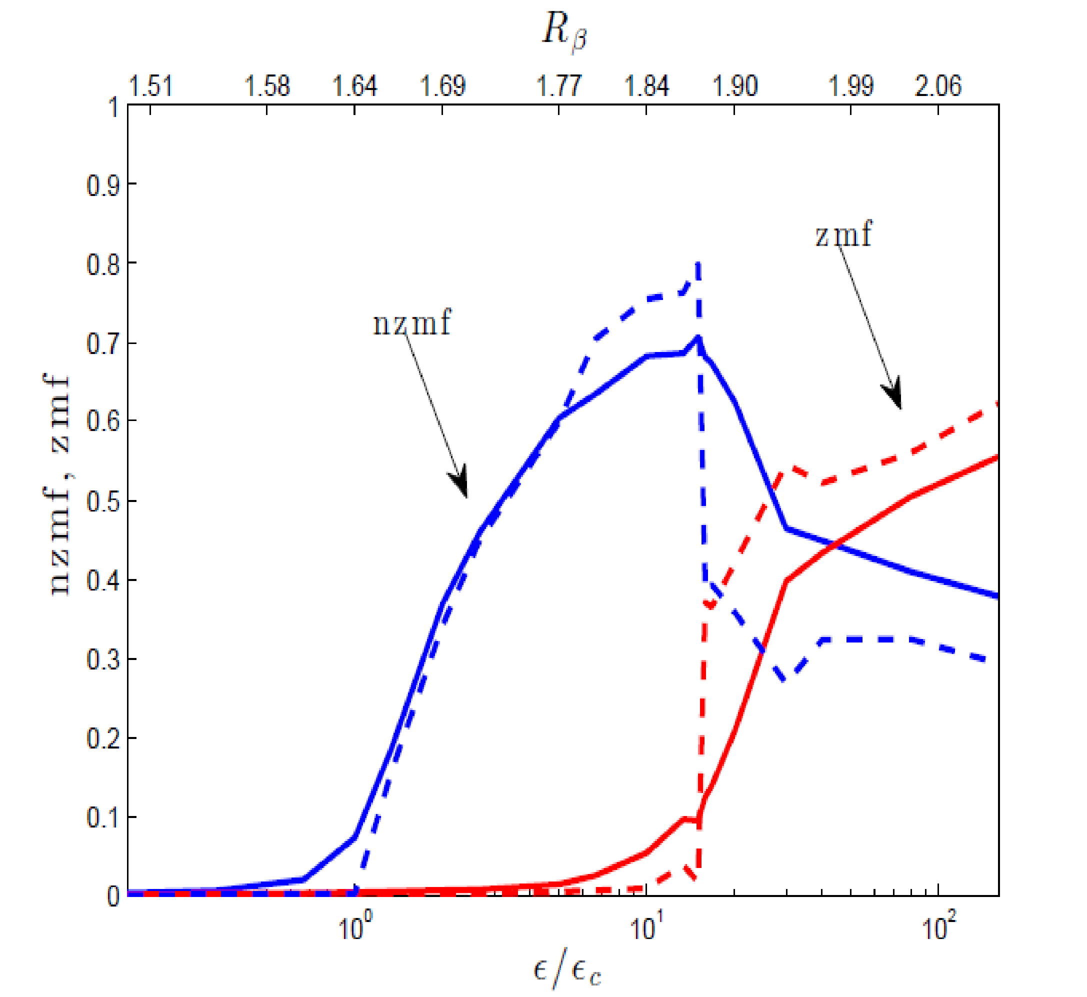
$(n_x, n_y) = (1, 5)$



# Accurate prediction of scale, amplitude

$$\mathrm{zmf} = \frac{\sum_{l:l<K_f} \hat{E}(k=0,l)}{\sum \hat{E}(k,l)}$$

$$\mathrm{nzmf} = \frac{\sum_{k,l:K<K_f} \hat{E}(k,l)}{\sum \hat{E}(k,l)} - \mathrm{zmf}$$

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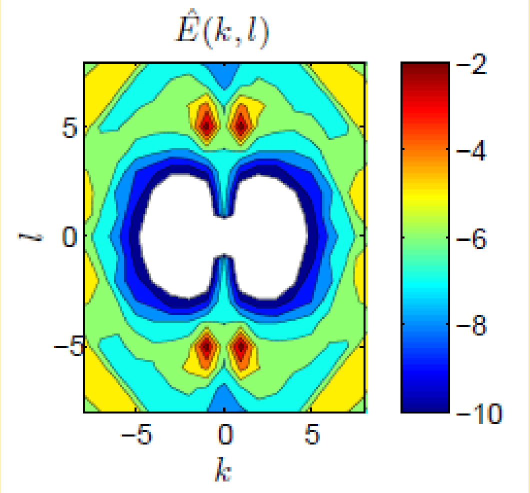
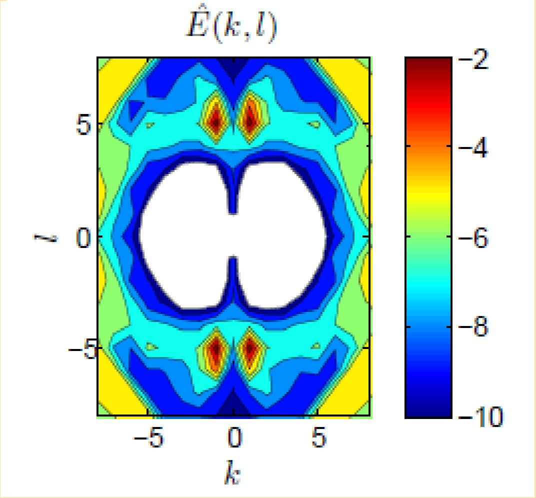
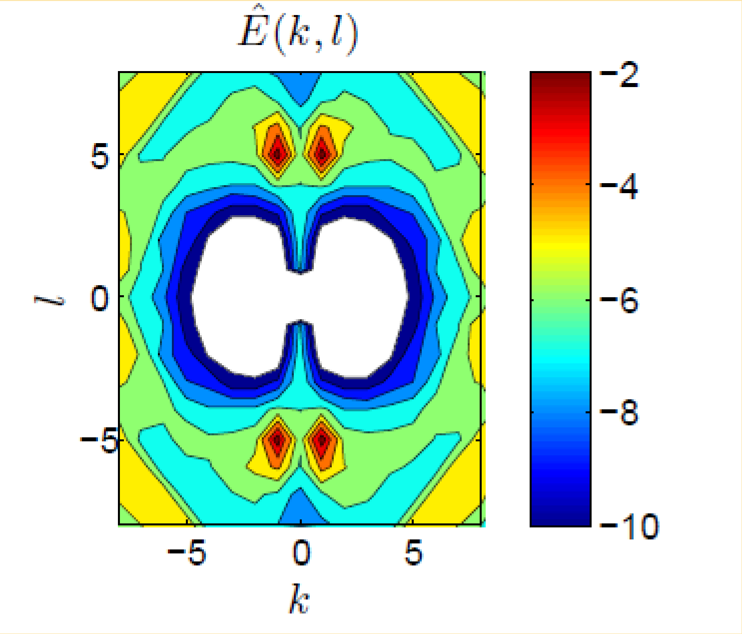



S3T

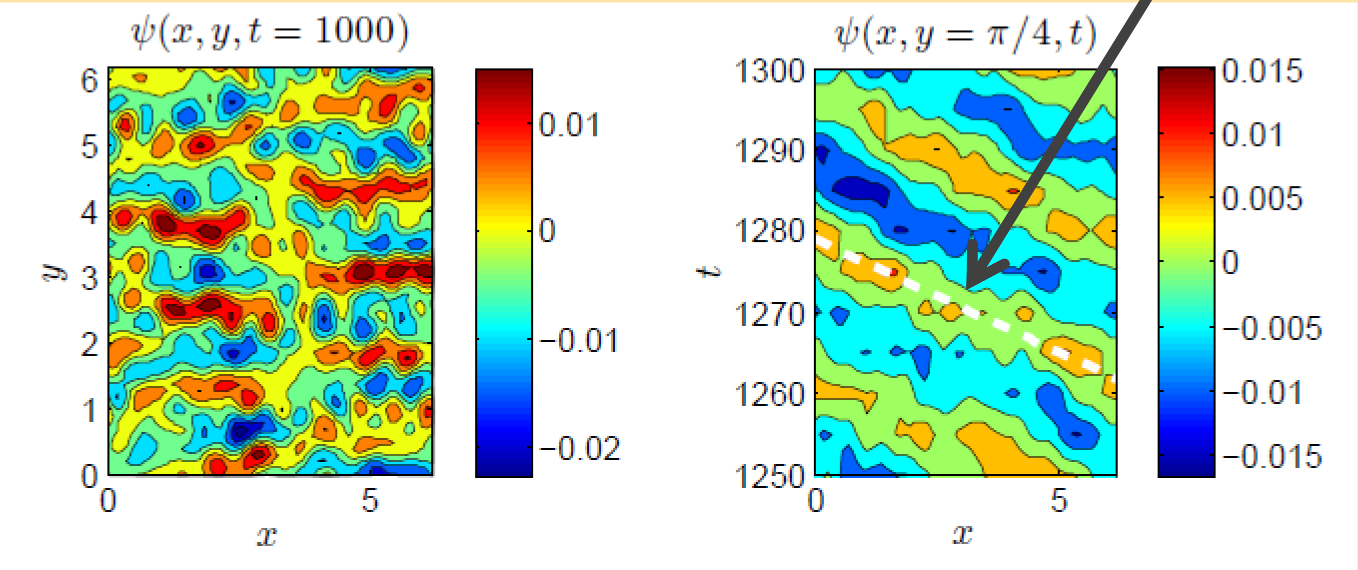


# Accurate prediction of phase speed

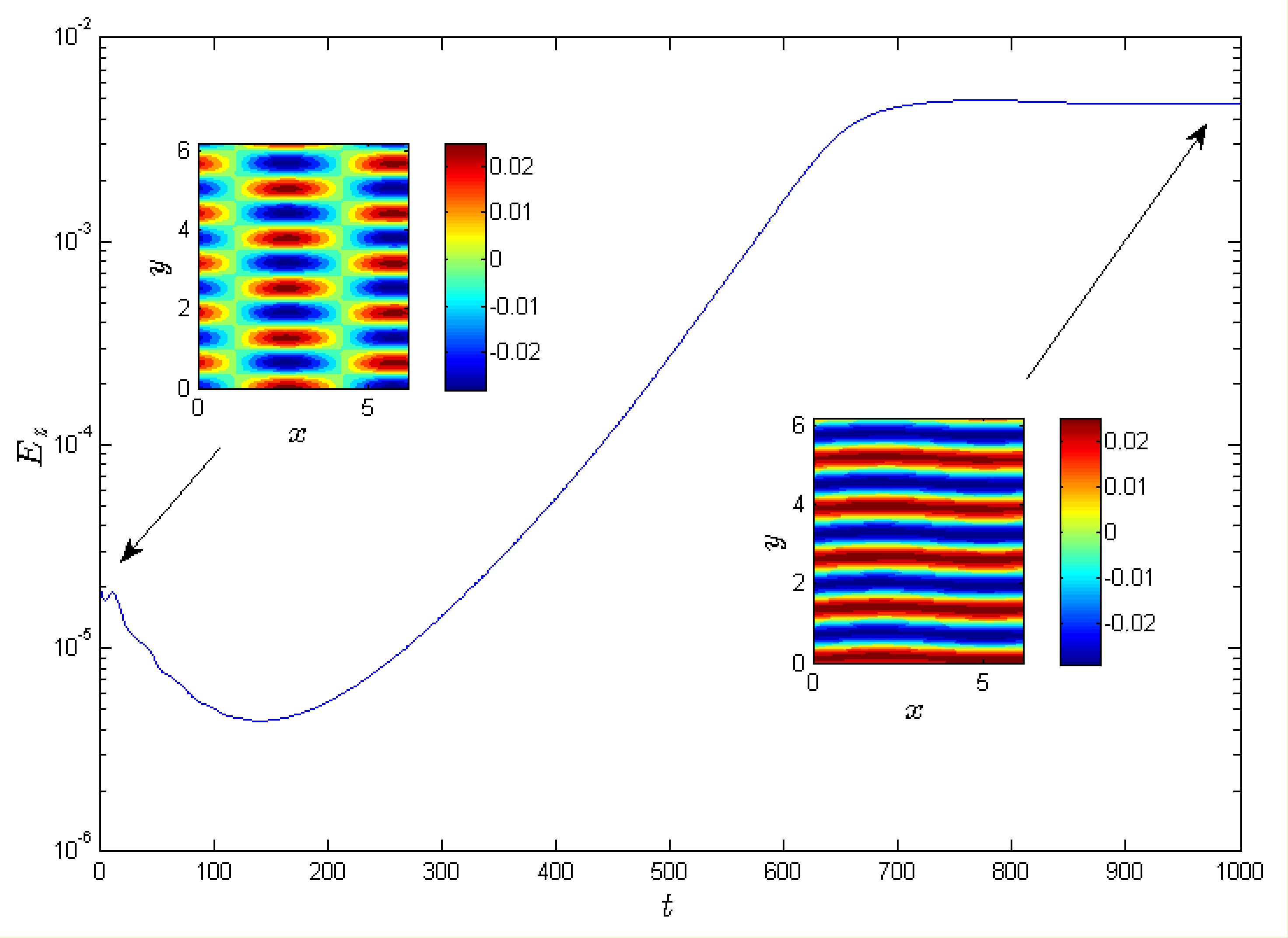
$\varepsilon/\varepsilon_c = 4$




phase speed of equilibrated S3T structure



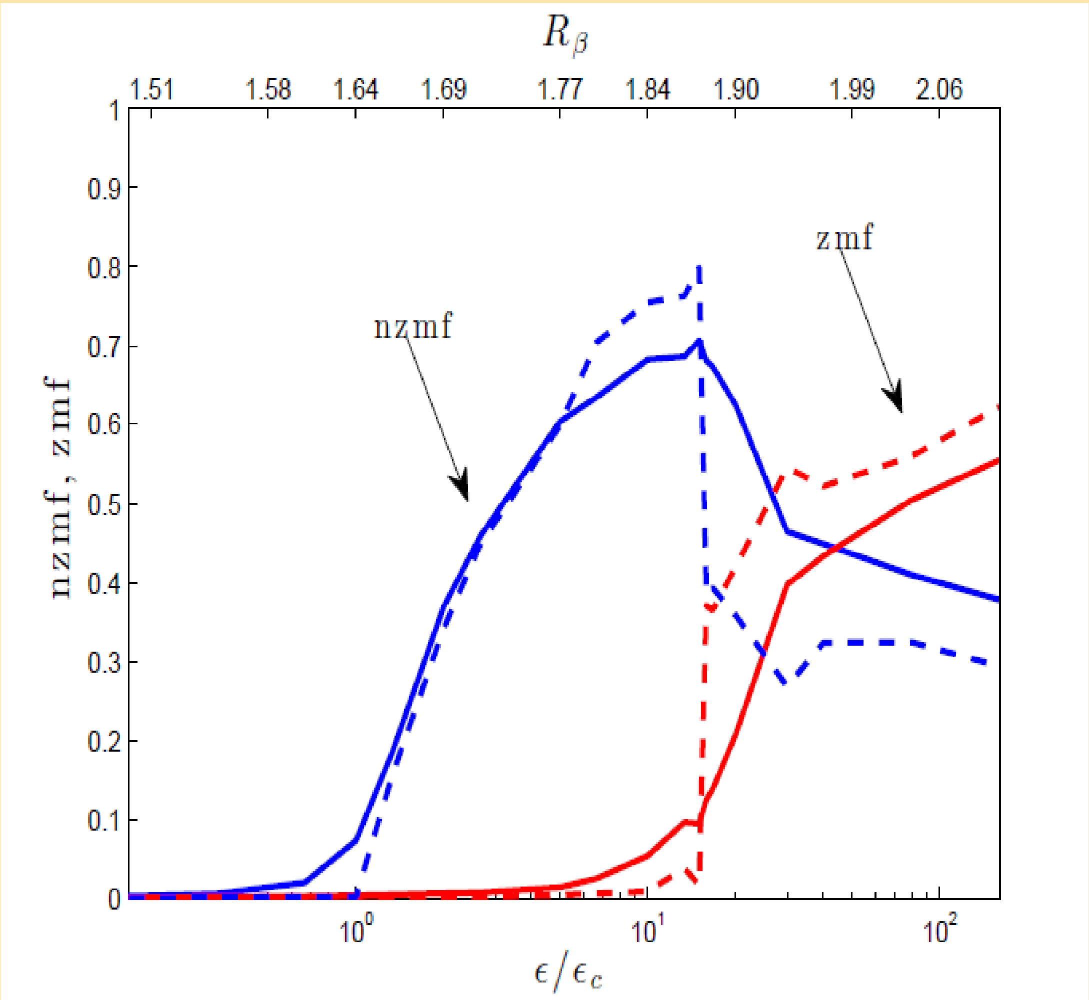
# 2nd transition: the traveling waves become unstable

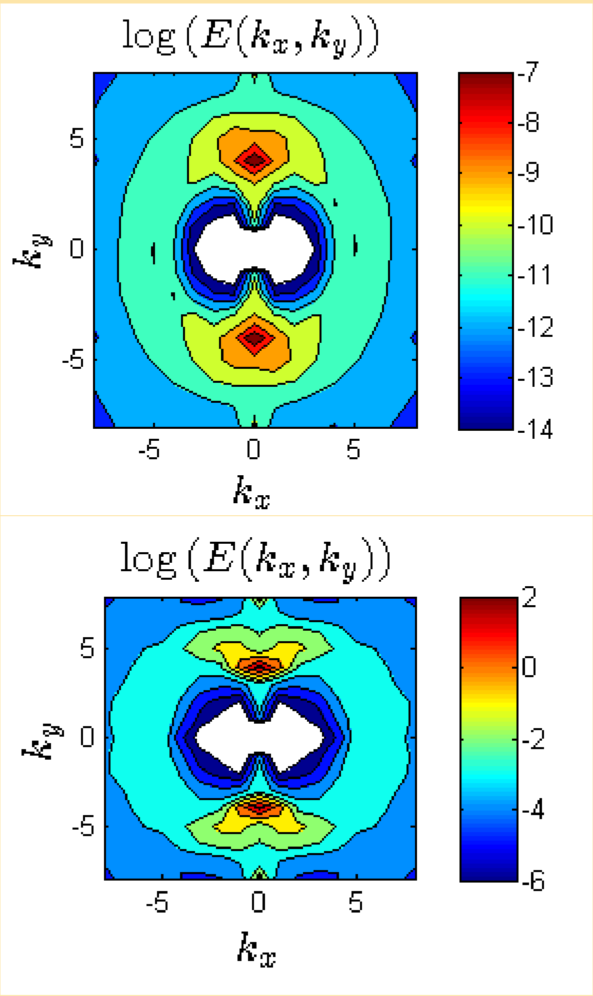



the finite amplitude traveling wave states become unstable to zonal jets

# Accurate prediction of the second transition

$$\text{zmf} = \frac{\sum_{l:l<K_f} \hat{E}(k=0,l)}{\sum \hat{E}(k,l)} \qquad \text{nzmf} = \frac{\sum_{k,l:K<K_f} \hat{E}(k,l)}{\sum \hat{E}(k,l)} - \text{zmf}$$



# Our goal

Develop a theory that accurately predicts:

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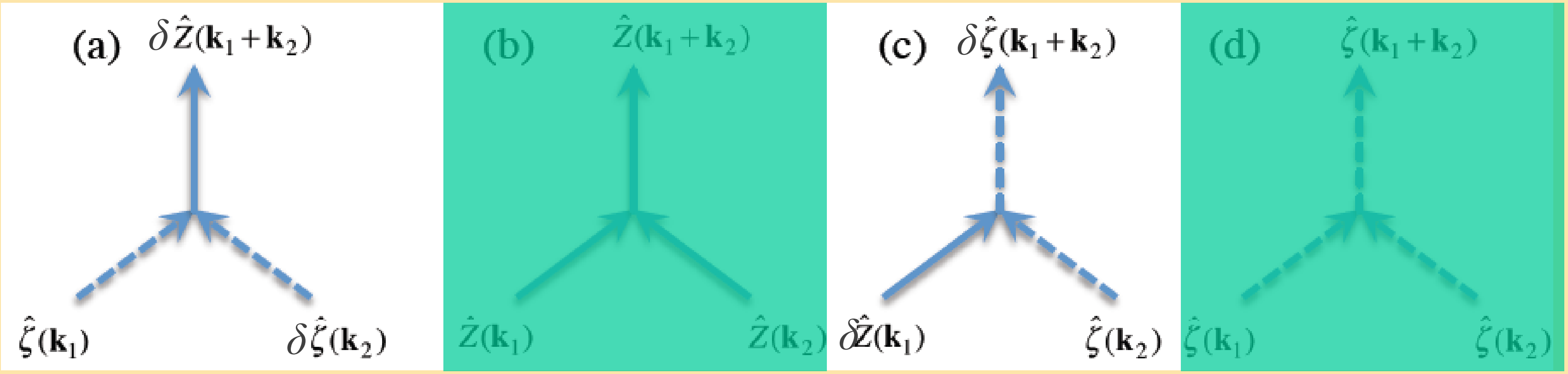
- The characteristics (scale, amplitude, phase speed) of the emergent structures

- Provides an explanation for the dynamics underlying structure formation

# Eddy-mean flow dynamics underlying instability




stability around
no mean flow

S3T

# Wave-mean flow dynamics underlying jet emergence

$$\left.\begin{aligned} \frac{dC}{dt} &= \left(A_1 + A_2\right)C + \Xi = 0 \\ \frac{dZ}{dt} &= -\left(U\partial_x + V\partial_y\right)Z - \beta V - rZ + F(C) = 0 \end{aligned}\right\} \rightarrow Z^E = 0, C^E = \Xi/2r$$

- We change the mean flow by $\delta Z$ and assume that the change is slow enough that the eddies are in equilibrium with the mean flow

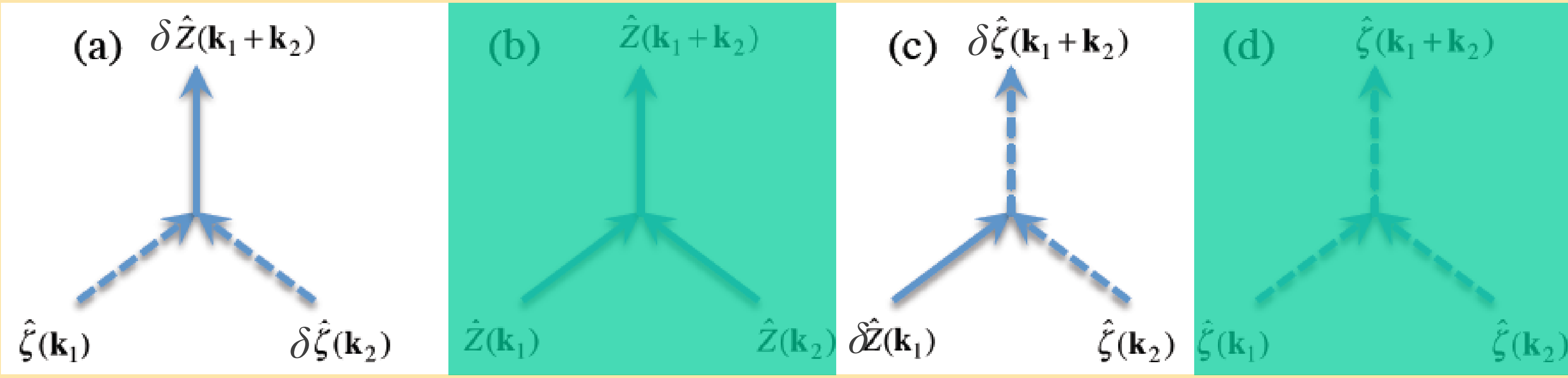
$$\left(A_1(\delta Z) + A_2(\delta Z)\right)C^{\mathrm{E}} + \left(A_1(Z^{\mathrm{E}}) + A_2(Z^{\mathrm{E}})\right)\delta C = 0 \rightarrow \delta C = g(\delta Z)$$

$$-\partial_x \delta\langle u'\zeta'\rangle - \partial_y \delta\langle v'\zeta'\rangle = f\left(\delta Z\right)$$

$$f(\delta Z) = \int\int \frac{d^2k}{(2\pi)^2} \frac{|\mathbf{k}\times\mathbf{n}|^2(k_s^2-k^2)(k^2-n^2)}{k^4 k_s^2 n^2[2-i(\omega_k+\omega_n-\omega_{k+n})]} \frac{\hat{\Xi}(\mathbf{k})}{2}$$

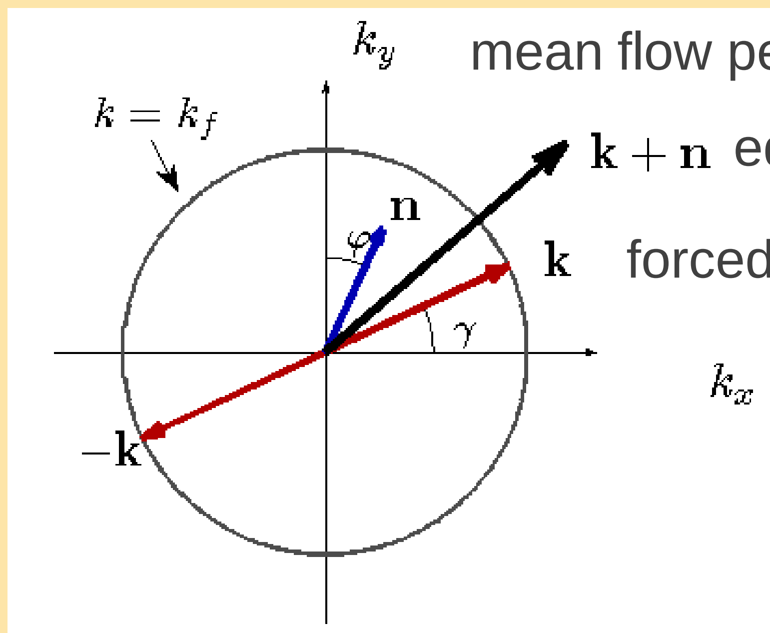
- Study the contribution of each forced wave in the flux divergence

# Eddy-mean flow dynamics underlying instability




stability around
no mean flow

S3T



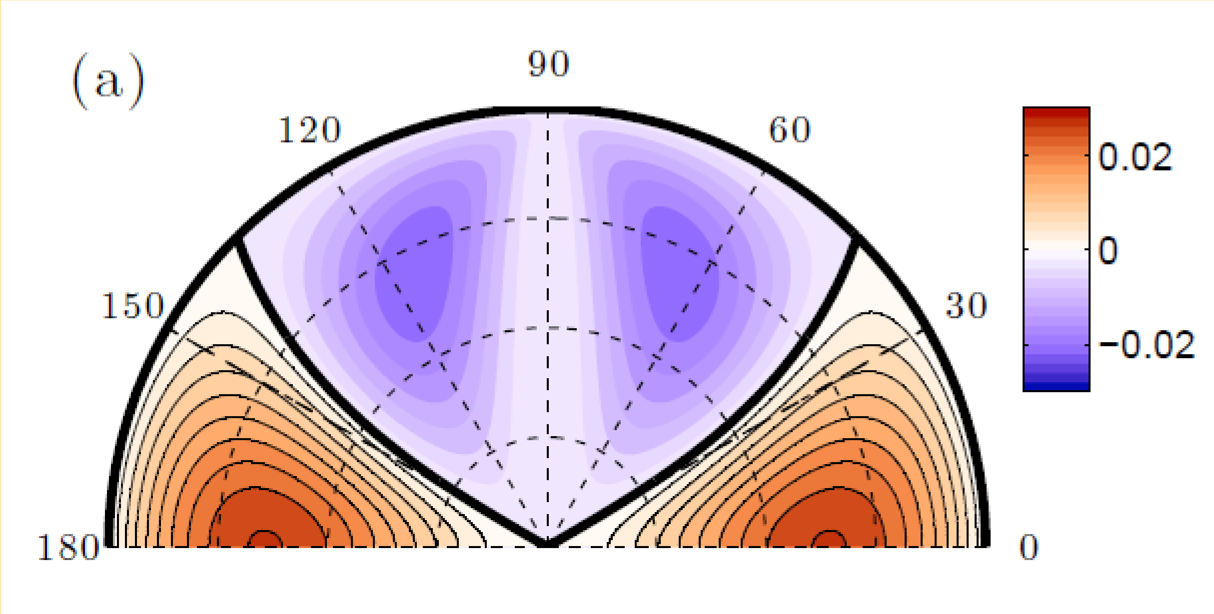

mean flow perturbation

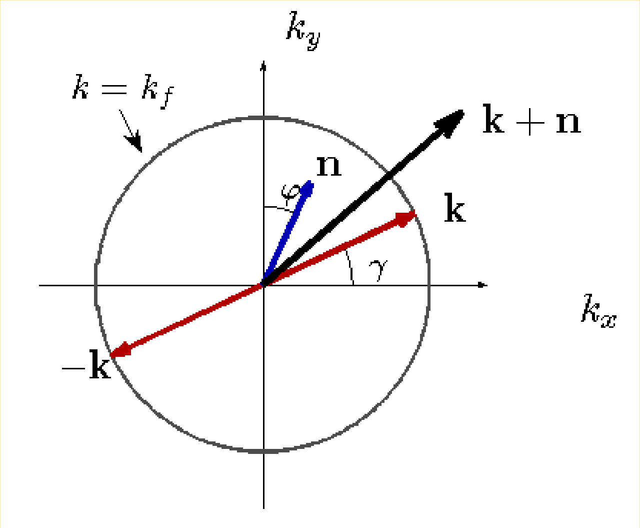
$\mathbf{k}+\mathbf{n}$ eddy perturbation

$\mathbf{k}$ forced eddy

$k_x$

# Which of the eddies matter ? ($\beta$<<$1$ limit)

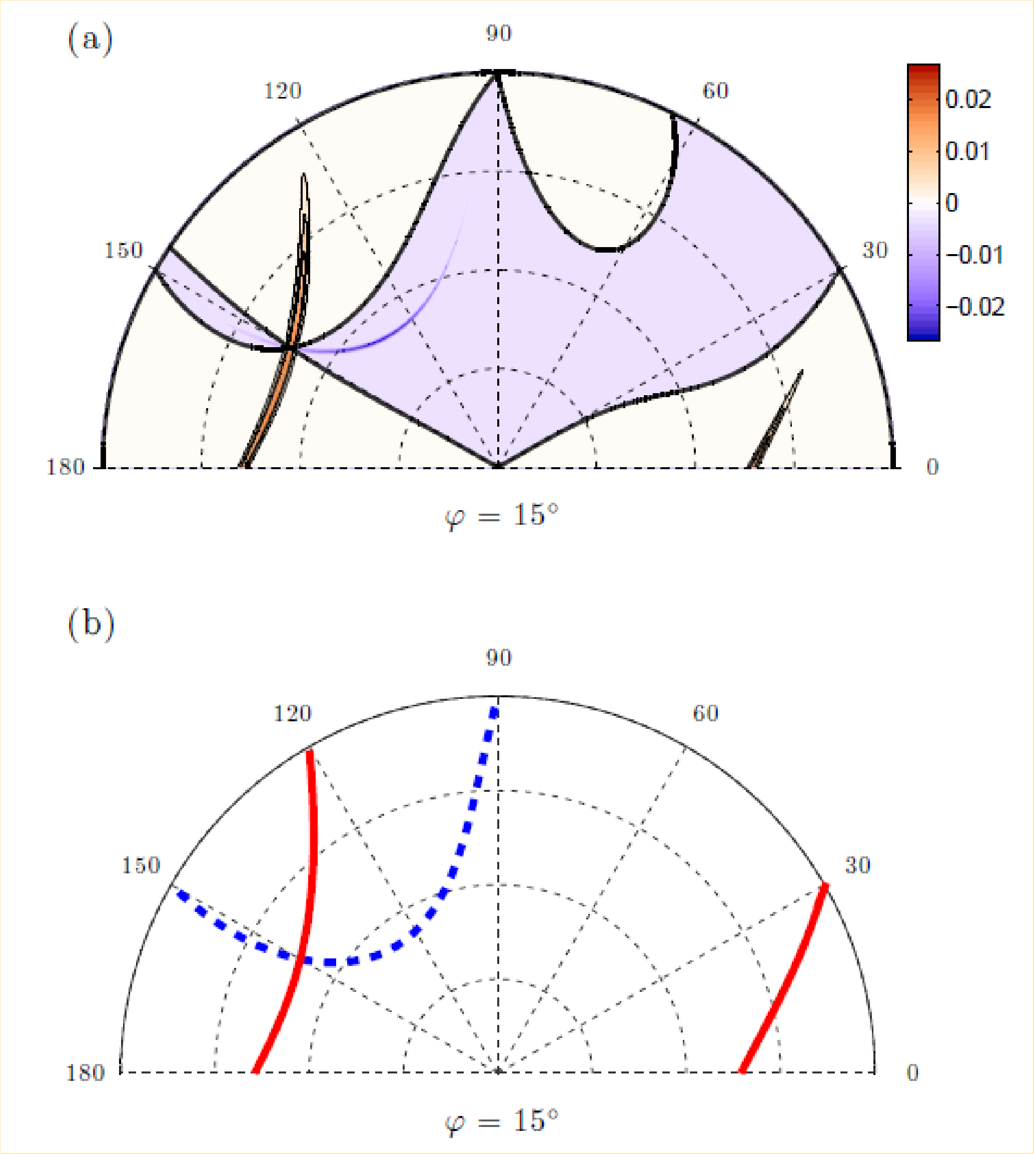



- Fluxes are determined by the sum of the effect of a broad band of eddies
- Orr dynamics (no time to show)
- Derive asymptotic expressions for the fluxes $f(\delta Z) = \beta^2 \frac{n^4}{64}(2+\cos(2\phi)) + O(\beta^4)$

  (negative hyper-diffusion)

# Which of the eddies matter ? ($\beta$>>1 limit)




- a narrow band of eddies
- Satisfy the near resonant condition $\omega_k+\omega_n-\omega_{k+n}$~$O(1/\beta)$
- Modulational instability

  (but in a forced-dissipative turbulent flow !!!!)

# Take home messages…

Using S3T, we are able to accurately predict:

- The regime transitions in the flow with the emergence of non-zonal westward propagating coherent structures and the emergence of zonal jets

- The scale, amplitude and phase speed of the emergent coherent structures in the turbulent flow

Using S3T we were able to study in detail the eddy-mean flow dynamics underlying the instability

S3T is a powerful tool to study bifurcations in turbulence and do stability theory for the cooperative interaction between turbulence and mean structures

# Thank you !


Bakas & Ioannou, 2013 : On the mechanism underlying the spontaneous emergence of barotropic zonal jets. *JAS*, **70**, 2251-2271

Bakas & Ioannou, 2013 : Emergence of large scale structure in planetary turbulence. *PRL,* **110***,* 224501

Bakas & Ioannou, 2014 : A theory for the emergence of coherent structures in beta-plane turbulence. *JFM,* **740***,* 312-341

Bakas, Constantinou & Ioannou, 2015 : S3T stability of the homogeneous state of barotropic beta-plane turbulence. *JAS*, (in press)