Emergence of large scale structures in barotropic turbulence

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Turbulent flows are organized into vortices and jets

banded jets  
Great Red Spot  
ocean jets & rings

(Richards et al 2006)
Simplest model: barotropic flow on a beta-plane

- Simplest setting
  - non-divergent flow in a doubly periodic β-plane channel

\[ \zeta = \partial_x v - \partial_y u = \Delta \psi \]
\[ \left( \partial_t + \bar{u} \cdot \nabla \right) \zeta + \beta v = -r \zeta - \nu \nabla^4 \zeta + \zeta(t) \]

- Spatially homogeneous forcing that is delta-correlated in time

\[ \langle \zeta(x_1, y_1, t_1) \zeta(x_2, y_2, t_2) \rangle = \delta(t_1 - t_2) \Xi(x_1, x_2, y_1, y_2) \]
\[ \Xi(x_1, x_2, y_1, y_2) = \sum \hat{\Xi}(k, l) e^{ik(x_1-x_2)+il(y_1-y_2)} \]

- Isotropic forcing injecting energy at rate $\epsilon$ in a narrow ring at $K_f$

\[ K_f = 10, \; \Delta K_f = 1, \; \beta = 10, \; r = 0.01, \; \nu = 10^{-6} \]
Two regime transitions in the flow

\[
zmf = \frac{\sum_{l \leq K_f} \hat{E}(k = 0, l)}{\sum \hat{E}(k, l)}
\]

\[
nzmf = \frac{\sum_{k : l \leq K_f} \hat{E}(k, l)}{\sum \hat{E}(k, l)} - zmf
\]
Two regime transitions in the flow

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\]
Non-zonal westward propagating coherent structures

\[ \frac{\varepsilon}{\varepsilon_c} = 4 \]
\[ R_\beta = 1.62 \]

peak at \((k_x, k_y) = (1, 5)\)

Rossby wave phase speed

\[ \log(E(k_x, k_y)) \]

\[ \psi(x, y, t = 1200) \]

\[ \psi(x, y = \pi/4, t) \]

\( a (k_x, k_y) = (1, 5) \) structure
Two regime transitions in the flow

\[
\text{zmf} = \frac{\sum_{l \leq K_f} \hat{E}(k=0,l)}{\sum \hat{E}(k,l)}
\]

\[
\text{nzmf} = \frac{\sum_{k,l : k \leq K_f} \hat{E}(k,l)}{\sum \hat{E}(k,l)} - \text{zmf}
\]

![Graph showing zmf and nzmf functions against \( R_{\beta} \) and \( \varepsilon / \varepsilon_c \).]
Zonal jets emerge, NZCS persist but slow down

\[ \frac{\varepsilon}{\varepsilon_c} = 50 \]
\[ R_\beta = 1.84 \]

slower than Rossby wave
Develop a theory that accurately predicts:

- The regime transitions in the flow (NZCS, jet emergence)
- The characteristics (scale, amplitude, phase speed) of the emergent structures
- Provides an explanation for the dynamics underlying structure formation
Theory for the statistical state dynamics: S3T

- Stochastic Structural Stability Theory (S3T)
  Farrell & Ioannou 2003

  Related: Cumulant Expansion (CE2) Marston et al. 2008

- Variables: mean + deviation: \( \zeta = Z + \zeta' \)

\[
\left( \partial_t + U \partial_x + V \partial_y \right) \zeta' + \left( \beta + Z_y \right) v' + Z_x u' = \xi - r \zeta' - v \nabla^4 \zeta' + \left\langle \tilde{u}' \cdot \nabla \zeta' \right\rangle - \tilde{u}' \cdot \nabla \zeta' \\
\left( \partial_t + U \partial_x + V \partial_y \right) Z + \beta V = -\partial_x \langle u' \zeta' \rangle - \partial_y \langle v' \zeta' \rangle - rZ - v \nabla^4 Z
\]
Cumulant expansion

\[
\left( \partial_t + U \partial_x + V \partial_y \right) Z + \beta V = -\partial_x \langle u' \zeta' \rangle - \partial_y \langle v' \zeta' \rangle - rZ - v \nabla^4 Z
\]

evolution of 1st cumulant \( Z = \langle \zeta \rangle \)

\[
-\partial_x \langle u' \zeta' \rangle - \partial_y \langle v' \zeta' \rangle = F(C)
\]

\[
C = \langle \zeta'_1 \zeta'_2 \rangle, \quad \zeta'_i = \zeta(x_i, y_i, t)
\]

\[
\frac{dC}{dt} = (A_1 + A_2) C + \Xi + G\left( \langle \zeta'_1 \zeta'_2 \zeta'_3 \rangle \right)
\]

\[
\frac{dZ}{dt} = -\left( U \partial_x + V \partial_y \right) Z - \beta V - rZ - v \nabla^4 Z + F(C)
\]

\[
\partial_t \zeta'_i = A_i \zeta'_i + \xi_i + \langle \vec{u}' \cdot \nabla \zeta'_i \rangle - \vec{u}' \cdot \nabla \zeta'_i
\]

\[
A_i = -\left( U_i \partial_{x_i} + V_i \partial_{y_i} \right) - \left( \beta + Z_{y_i} \right) \partial_{x_i} \nabla^{-2} + Z_{x_i} \partial_{x_i} \nabla^{-2} - r - v \nabla^4
\]
Second order closure of cumulant expansion

CE2: \[ G\left(\langle \zeta_1' \zeta_2' \zeta_3' \rangle\right) = 0 \iff \langle \bar{u}_i' \cdot \nabla \zeta_i' \rangle - \bar{u}_i' \cdot \nabla \zeta' = 0 \] ignore the eddy-eddy

S3T: \[ G\left(\langle \zeta_1' \zeta_2' \zeta_3' \rangle\right) = \Gamma - r_{eff} C \iff \langle \bar{u}_i' \cdot \nabla \zeta_i' \rangle - \bar{u}_i' \cdot \nabla \zeta' = \gamma(t) - r_{eff} \zeta' \]
eddy-eddy term acting as stochastic forcing and dissipation

• Ergodic assumption: ensemble average = time average

\[
\begin{align*}
\frac{dZ}{dt} &= -\left(U \partial_x + V \partial_y\right)Z - \beta V - rZ - \nu \nabla^4 Z + F(C) \\
\frac{dC}{dt} &= (A_1 + A_2)C + \Xi
\end{align*}
\]
Closed, deterministic system for the coherent flow & eddy statistics
Regime transitions in turbulence as an instability

\[
\frac{dC}{dt} = (A_1 + A_2) C + \Xi = 0
\]
\[
\frac{dZ}{dt} = -\left( U \partial_x + V \partial_y \right) Z - \beta V - rZ + F(C) = 0
\]
\[\rightarrow Z^E, C^E\]

• Linearization: study the evolution of small perturbations in the mean structure \( \delta Z \) and in the eddy statistics \( \delta C \):

\[
Z^E = 0, C^E = \Xi / 2r
\]
\[
\frac{d}{dt} \begin{pmatrix} \delta Z \\ \delta C \end{pmatrix} = L(Z^E, C^E) \begin{pmatrix} \delta Z \\ \delta C \end{pmatrix}
\]

intensify the mean structure through upgradient vorticity fluxes

turbulent eddies
organizes the eddies so that the eddy fluxes are reinforced

mean structure
Stability of homogeneous equilibrium

\[ Z^E = 0, C^E = \Xi / 2r \]
Critical curve

- Zonal jets more unstable
- Non-zonal structures more unstable
- Jupiter
- Atmosphere
- Ocean

Variables:
- $\epsilon_c^*$
- $\beta^*$

Equations:
- $\beta^{-2}$
- $\beta^{*2}$
- $\beta^{*1/2}$
S3T predicts the first regime transition

\[
zmf = \frac{\sum_{l: l < K_f} \hat{E}(k = 0, l)}{\sum \hat{E}(k, l)}
\]

\[
nzmf = \frac{\sum_{k: k < K_f} \hat{E}(k, l)}{\sum \hat{E}(k, l)} - zmf
\]
Most unstable structures

stationary zonal jets

propagating non-zonal structures
Equilibration of instabilities: emergent structures

\[ \tilde{\beta} = 100 \]
\[ \tilde{\varepsilon} / \varepsilon_c = 4 \]
\[ R_\beta = 1.62 \]

\( (n_x, n_y) = (1, 5) \)
Accurate prediction of scale, amplitude

\[
zmf = \frac{\sum_{l: l \leq k_f} \hat{E}(k=0,l)}{\sum \hat{E}(k,l)}
\]

\[
nzmf = \frac{\sum_{k,l: k \leq k_f} \hat{E}(k,l)}{\sum \hat{E}(k,l)} - zmf
\]
Accurate prediction of phase speed

$\varepsilon / \varepsilon_c = 4$

phase speed of equilibrated S3T structure
2\textsuperscript{nd} transition: the traveling waves become unstable

the finite amplitude traveling wave states become unstable to zonal jets
Accurate prediction of the second transition

\[
zmf = \frac{\sum_{l:k \leq k_f} \hat{E}(k=0,l)}{\sum \hat{E}(k,l)}
\]

\[
zmf = \frac{\sum \hat{E}(k,l)}{\sum \hat{E}(k,l)} - zmf
\]
Develop a theory that accurately predicts:

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- Provides an explanation for the dynamics underlying structure formation
Eddy-mean flow dynamics underlying instability

(a) $\delta \hat{Z}(k_1 + k_2)$
(b) $\hat{Z}(k_1 + k_2)$
(c) $\delta \hat{\zeta}(k_1 + k_2)$
(d) $\hat{\zeta}(k_1 + k_2)$

Stability around no mean flow

S3T
Wave-mean flow dynamics underlying jet emergence

\[
\frac{dC}{dt} = \left( A_1 + A_2 \right) C + \Xi = 0
\]
\[
\frac{dZ}{dt} = -\left( U \partial_x + V \partial_y \right) Z - \beta V - rZ + F(C) = 0
\]
\[
\to Z^E = 0, C^E = \Xi / 2r
\]

• We change the mean flow by \( \delta Z \) and assume that the change is slow enough that the eddies are in equilibrium with the mean flow

\[
\left( A_1 (\delta Z) + A_2 (\delta Z) \right) C^E + \left( A_1 (Z^E) + A_2 (Z^E) \right) \delta C = 0 \to \delta C = g(\delta Z)
\]

\[
-\partial_x \delta \langle u' \zeta' \rangle - \partial_y \delta \langle v' \zeta' \rangle = f(\delta Z)
\]

\[
f(\delta Z) = \int \int \frac{d^2 k}{(2\pi)^2} \frac{|k \times n|^2 (k_s^2 - k_0^2) (k^2 - n^2)}{k^4 k_s^2 n^2 [2 - i(\omega_k + \omega_n - \omega_{k+n})]} \Xi(k)
\]

• Study the contribution of each forced wave in the flux divergence
Eddy-mean flow dynamics underlying instability

stability around no mean flow

S3T

mean flow perturbation

eddy perturbation

forced eddy
Which of the eddies matter? ($\beta << 1$ limit)

- Fluxes are determined by the sum of the effect of a broad band of eddies.
- Orr dynamics (no time to show).
- Derive asymptotic expressions for the fluxes:
  
  \[
  f(\delta Z) = \beta^2 \frac{n^4}{64} (2 + \cos(2\phi)) + O(\beta^4)
  \]

(negative hyper-diffusion)
Which of the eddies matter? ($\beta >> 1$ limit)

- a narrow band of eddies
- Satisfy the near resonant condition $\omega_k + \omega_n - \omega_{k+n} \sim O(1/\beta)$
- Modulational instability

(but in a forced-dissipative turbulent flow !!!!)
Take home messages…

Using S3T, we are able to accurately predict:

- The regime transitions in the flow with the emergence of non-zonal westward propagating coherent structures and the emergence of zonal jets
- The scale, amplitude and phase speed of the emergent coherent structures in the turbulent flow

Using S3T we were able to study in detail the eddy-mean flow dynamics underlying the instability

S3T is a powerful tool to study bifurcations in turbulence and do stability theory for the cooperative interaction between turbulence and mean structures
Thank you!

Bakas & Ioannou, 2013: On the mechanism underlying the spontaneous emergence of barotropic zonal jets. *JAS*, **70**, 2251-2271

Bakas & Ioannou, 2013: Emergence of large scale structure in planetary turbulence. *PRL*, **110**, 224501
