# Rare event in geophysical fluid dynamics and large deviation theory

F. BOUCHET (CNRS) – ENS-Lyon and CNRS

March 2015 – Les Houches

ENS DE LYON

European Research Council
Established by the European Commission

# Outline

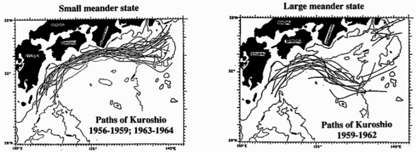
1. Rare events may matter in geophysical fluid dynamics
   - Rare transitions between two attractors
   - Rare events that have a huge impact
2. Large deviation theory and rare transitions
   - Large deviation theory for dynamical systems
   - Large deviations in the weak noise regime (Freidlin–Wentzell)
3. Numerical computation of rare events
   - Computing numerically action minima (instantons)
   - Adaptive multilevel splitting: an example of rare event algorithm based on selection and cloning

# Outline

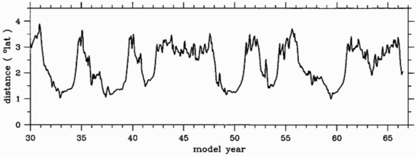
1. Rare events may matter in geophysical fluid dynamics
   - Rare transitions between two attractors
   - Rare events that have a huge impact
2. Large deviation theory and rare transitions
   - Large deviation theory for dynamical systems
   - Large deviations in the weak noise regime (Freidlin–Wentzell)
3. Numerical computation of rare events
   - Computing numerically action minima (instantons)
   - Adaptive multilevel splitting: an example of rare event algorithm based on selection and cloning

# Kuroshio Bistability

## An example where rare transitions matter in ocean dynamics




M. J. Schmeits and H. A. Dijkstra (adapted from Taft)



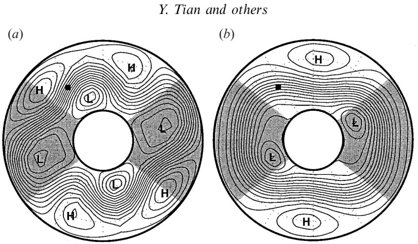

Primitive equation model - Qiu and Miao (2000)

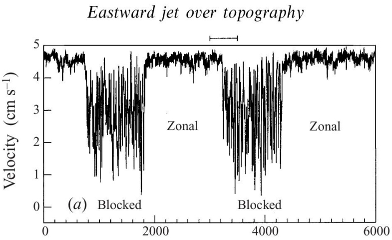
- Are two successive transitions statistically independent ? Can we compute the transition times (or transition rate)?

# Rare Transitions in Rotating Tank Experiments

## The rotation as an ordering field (Quasi Geostrophic dynamics)

### Transitions between blocked and zonal states

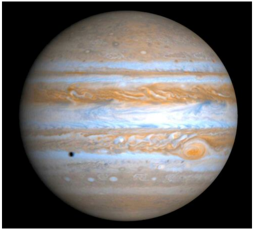


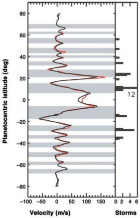

*Y. Tian and col, J. Fluid. Mech. (2001)* (groups of H. Swinney and M. Ghil)

# Jupiter's Zonal Jets

## We look for a theoretical description of zonal jets



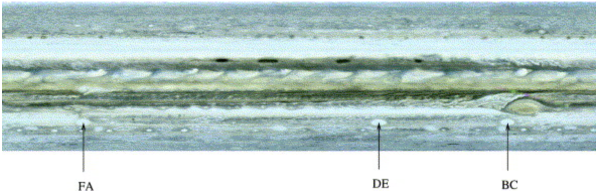
Jupiter's atmosphere

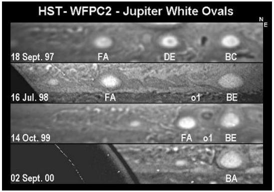



Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

# Have we Lost One of Jupiter's Jets?

## What is the probability of this event ?

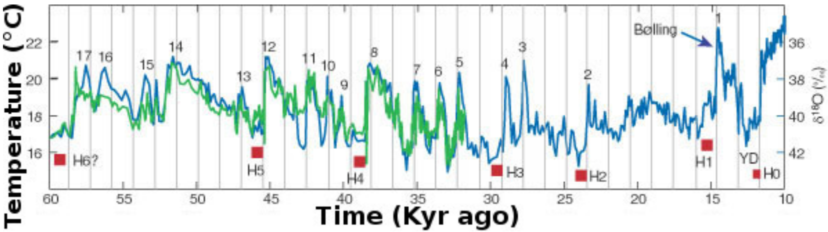



Jupiter's white ovals (see Youssef and Markus 2005)

The white ovals appeared in 1939-1940 (Rogers 1995). Following an instability of the zonal jet?

# Abrupt Climate Changes
## Long times matter

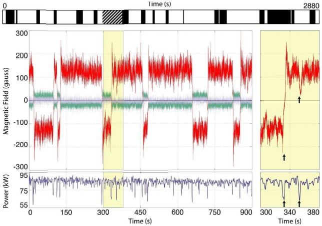



Temperature versus time: Dansgaard–Oeschger events (S. Rahmstorf)

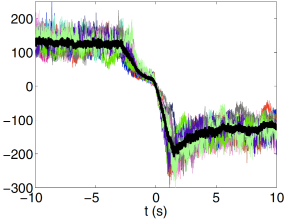
- What is the dynamics and probability of abrupt climate changes?

# Random Transitions in Turbulence Problems

## Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)




Magnetic field timeseries




Zoom on reversal paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another (reactive paths) often occur through a predictable path.

# The Main Scientific Issues

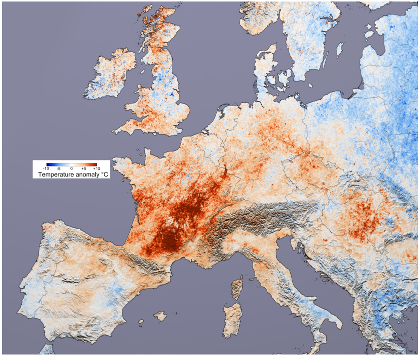
- How to characterize and predict the attractors of turbulent geophysical flows?
- In case of multiple attractors, can we compute their relative probability?
- Can we compute the transition paths and the transition rates?
- For most geophysical problems, an approach through direct numerical simulation is impossible (trade off between realistic turbulence representation and physical time - here one need both).
- Can we devise new theoretical and numerical tools to tackle these issues?

# Outline

1. Rare events may matter in geophysical fluid dynamics
   - Rare transitions between two attractors
   - Rare events that have a huge impact
2. Large deviation theory and rare transitions
   - Large deviation theory for dynamical systems
   - Large deviations in the weak noise regime (Freidlin–Wentzell)
3. Numerical computation of rare events
   - Computing numerically action minima (instantons)
   - Adaptive multilevel splitting: an example of rare event algorithm based on selection and cloning

# Extreme Heat Waves

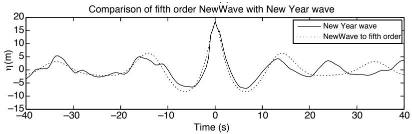
## Example: the 2003 heat wave over western Europe





July 20 2003-August 20 2003 land surface temperature minus the average for the same period for years 2001, 2002 and 2004 (TERRA MODIS).

# Rogue Waves




Draupner rogue wave and Walker et al. large deviation theory.

Walker et al (2004) (see also a nice account by Oliver Buhler (2007)).

# Rare Events with a Huge Impact

- Heat waves, rogue waves, floods, droughts, extreme precipitations, and so on

The scientific questions:

- What is the probability and the dynamics of those rare events?
- Is the dynamics leading to such rare events predictable?
- How to sample rare events, their probability, and their dynamics.
- Are direct numerical simulations a reasonable approach?
- Can we devise new theoretical and numerical tools to tackle these issues?

# Outline

1. Rare events may matter in geophysical fluid dynamics
   - Rare transitions between two attractors
   - Rare events that have a huge impact
2. Large deviation theory and rare transitions
   - Large deviation theory for dynamical systems
   - Large deviations in the weak noise regime (Freidlin–Wentzell)
3. Numerical computation of rare events
   - Computing numerically action minima (instantons)
   - Adaptive multilevel splitting: an example of rare event algorithm based on selection and cloning

# Large Deviation Theory for Dynamical Systems

- Large deviation theory is a general framework to describe probability distribution in asymptotic limits

$$P_\varepsilon[X=x] \underset{\varepsilon \ll 1}{\approx} C e^{-\frac{\mathscr{F}[x]}{\varepsilon}}.$$

- For equilibrium statistical mechanics, $\mathscr{F}$ is the free energy, and $\varepsilon = k_B T/N$.

Three main frameworks for large deviations for dynamical systems:

- Dynamical systems with small noises: Freidlin-Wentzell theory.
- Large deviations for time integrated observable: Donsker–Varadhan.
- Large deviations for the slow evolution of dynamical systems with two time scales (talk of Tomas Tangarife yesterday).

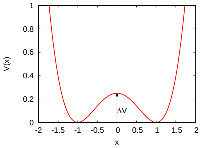
# Outline

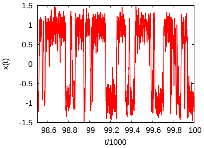
1. Rare events may matter in geophysical fluid dynamics
   - Rare transitions between two attractors
   - Rare events that have a huge impact
2. Large deviation theory and rare transitions
   - Large deviation theory for dynamical systems
   - Large deviations in the weak noise regime (Freidlin–Wentzell)
3. Numerical computation of rare events
   - Computing numerically action minima (instantons)
   - Adaptive multilevel splitting: an example of rare event algorithm based on selection and cloning

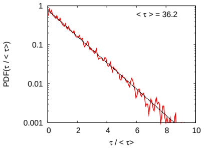
# Kramer's Problem: a Pedagogical Example for Bistability

**Historical example:** Computation by Kramer of Arrhenius' law for a bistable mechanical system with stochastic noise

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T_e}\eta(t) \text{ Rate : } \lambda = \frac{1}{\tau}\exp\left(-\frac{\Delta V}{k_B T_e}\right).$$




The problem was solved by Kramer (30'). Modern approach: path integral formulation (instanton theory, physicists) or large deviation theory (Freidlin–Wentzell, mathematicians).

# Path Integrals for ODE – Onsager Machlup (50')

- Path integral representation of transition probabilities:

$$P(x_t,T;x_0,0)=\int_{x(0)=x_0}^{x(T)=x_T} e^{-\frac{\mathscr{A}_T[x]}{2k_B T_e}}\mathscr{D}[x]$$

with $\mathscr{A}_T[x]=\int_0^T \mathscr{L}[x,\dot{x}]\ \mathrm{d}t$ and $\mathscr{L}[x,\dot{x}]=\frac{1}{2}\left[\dot{x}+\frac{\mathrm{d}V}{\mathrm{d}x}(x)\right]^2$.

- The most probable path from $x_0$ to $x_T$ is the minimizer of

$$A_T(x_0,x_T)=\min_{\{x(t)\}}\left\{\mathscr{A}_T[x]\,|x(0)=x_0 \text{ and } x(T)=x_T\right\}.$$

- We may consider the low temperature limit, using a saddle point approximation (WKB), Then we obtain the large deviation result

$$-2k_BT_e\log P(x_T,T;x_0,0)\underset{\frac{k_BT_e}{\Delta V}\to 0}{\sim}\min_{\{x(t)\}}\left\{\mathscr{A}_T[x]\,|x(0)=x_0 \text{ and } x(T)=x_T\right\}$$

# Most Transition Paths Follow the Instanton

- In the weak noise limit, most transition paths follow the most probable path (instanton)

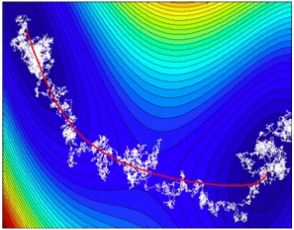


Figure by Eric Van den Eijnden

- For gradient dynamics, instantons are time reversed relaxation paths from a saddle to an attractor. Arrhenius law then follows

$$\log P(x_1, T; x_{-1}, 0) \underset{\frac{k_B T_e}{\Delta V}\to 0}{\sim} -\frac{\Delta V}{k_B T_e}.$$

# Non Gradient Dynamics and Transverse Decomposition

- Does this phenomenology remain true for non gradient dynamics? For instance for turbulent flows.

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t)$$

- Generically, there exists a transverse decomposition

$$\mathbf{b}(\mathbf{x}) = -\nabla V(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \text{ with for all } \mathbf{x},\ \nabla V(\mathbf{x}).\mathbf{G}(\mathbf{x}) = 0.$$

- Then

$$\log P(x_1, T; x_{-1}, 0) \sim \frac{\min_{\{x(t)|x(0)=x_0 \text{ and } x(T)=x_T\}} \{\mathscr{A}_T[x]\}}{2\varepsilon} = -\frac{\Delta V}{\varepsilon}.$$

- The phenomenology holds for most non-gradient dynamics. The difficulty is to compute $V$ (or $\mathbf{G}$).

# Outline


1. Rare events may matter in geophysical fluid dynamics
   - Rare transitions between two attractors
   - Rare events that have a huge impact
2. Large deviation theory and rare transitions
   - Large deviation theory for dynamical systems
   - Large deviations in the weak noise regime (Freidlin–Wentzell)
3. Numerical computation of rare events
   - Computing numerically action minima (instantons)
   - Adaptive multilevel splitting: an example of rare event algorithm based on selection and cloning

# Numerical Computation of Action Minima

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t).$$

- Action

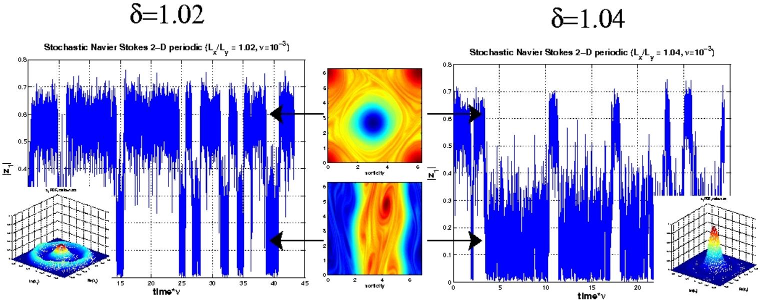
$$\mathscr{A}[\mathbf{x}] = \int_0^T \mathscr{L}[\mathbf{x},\dot{\mathbf{x}}]\ \mathrm{d}t \text{ and } \mathscr{L}[\mathbf{x},\dot{\mathbf{x}}] = \frac{1}{2}[\dot{\mathbf{x}} - \mathbf{b}(\mathbf{x})]^2.$$

- Numerical computation of action minima.

E. Vanden-Eijnden, W. E and W. Ren, (2004). E Vanden-Eijnden and M Heymann, (2008).
Tobias Grafke will discuss applications to the Burger equation this afternoon.

# Non-Equilibrium Phase Transition for the 2D Navier–Stokes Eq.

## The time series and PDF of the Order Parameter




Order parameter : $z_1 = \int dxdy \, \exp(iy)\omega(x,y)$.

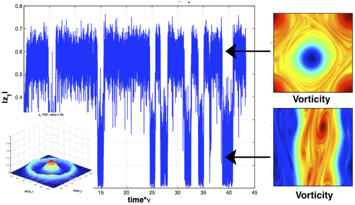
For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$

F. Bouchet and E. Simonnet, PRL, 2009.

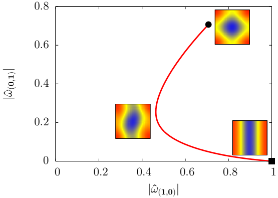
# Most Probable Transitions (Instantons) for the 2D Navier-Stokes Eq.

With J. Laurie

- Large deviation theory: instantons as minimum action paths.




2D Navier-Stokes equations (time: 10 000) (PRL)




Numerical instanton (time of order 1) (J. Stat. Phys.)

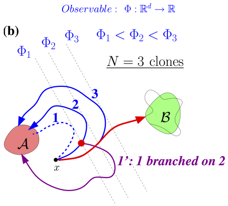
J. Laurie and F. Bouchet, Computation of rare transitions in the barotropic quasi-geostrophic equations, New J. Phys., 2014

# Outline

1. Rare events may matter in geophysical fluid dynamics
   - Rare transitions between two attractors
   - Rare events that have a huge impact
2. Large deviation theory and rare transitions
   - Large deviation theory for dynamical systems
   - Large deviations in the weak noise regime (Freidlin–Wentzell)
3. Numerical computation of rare events
   - Computing numerically action minima (instantons)
   - Adaptive multilevel splitting: an example of rare event algorithm based on selection and cloning

# Rare Events and Adaptive Multilevel Splitting

## AMS: an algorithm to compute rare events, for instance rare reactive trajectories




Probability estimate:

$$\hat{\alpha} = \prod P(l_k, l_{k+1})$$

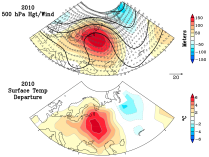
with $P(l_k, l_{k+1}) = (1 - 1/N)$

AMS algorithm

F. Cérou, A. Guyader. (2007) F. Cérou, A.Guyader, T. Lelièvre, and D. Pommier (2011).

Eric Simonnet will use this algorithm to compute rare transition for quasi-geostrophic zonal jets similar to Jupiter jets.

# Sampling Extreme Heat Waves using Large Deviation Algorithms (with J. Wouters and F. Ragone)




2010 Heat Wave over Eastern Europe (Dole and col., 2011)

- Computing the dynamical paths leading to extreme heat waves through rare event algorithms.
- We use the Planet Simulator (PlaSim) model (an Earth system model of intermediate complexity).

# GFD Problems Solved using Large Deviation Theory

Few more papers using large deviation theory and/or rare event algorithms in a Geophysical Fluid Dynamics context:

- P. H. Haynes and J. Vanneste, Dispersion in the large-deviation regime. Part 1: shear flows and periodic flows, J. Fluid Mech., 2014.
- P. H. Haynes and J. Vanneste, Dispersion in the large-deviation regime. Part 2: cellular flows at large Péclet number, J. Fluid Mech., 2014.
- J. G. Esler, Adaptive stochastic trajectory modeling in the chaotic advection regime, JFM, in press.
- F. Bouchet, and A. Venaille, Statistical mechanics of two-dimensional and geophysical flows, Physics Reports, 2012.
- F. Bouchet, J. Laurie and O. Zaboronsky, Langevin dynamics, large deviations and instantons for the quasi-geostrophic model,J. Stat. Phys., 2014.

## Summary and Perspectives

- Large deviation theory can be applied to geophysical turbulence.
- The dynamics leading to rare events is usually predictable, even for turbulent flows.
- With rare event algorithms, we can compute probability of rare events that can not be sampled using direct numerical simulations.
- This is a promising field that will help answering scientific issues that can not be adressed using classical theoretical or numerical tools.