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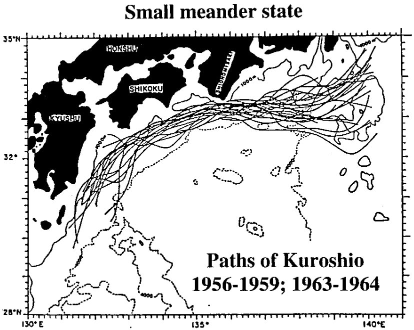
Predicting extreme events in fluid turbulence via large deviation minimizers

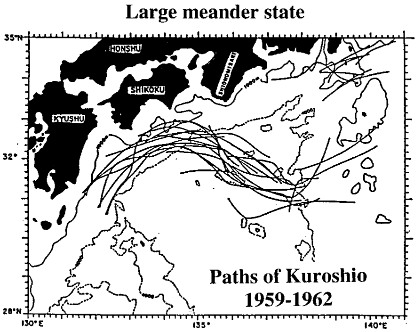
Tobias Grafke, E. Vanden-Eijnden, T. Schäfer, R. Grauer

Bi-Stability in fluid dynamics

Kuroshio stream

Bistability in Ocean current[1]



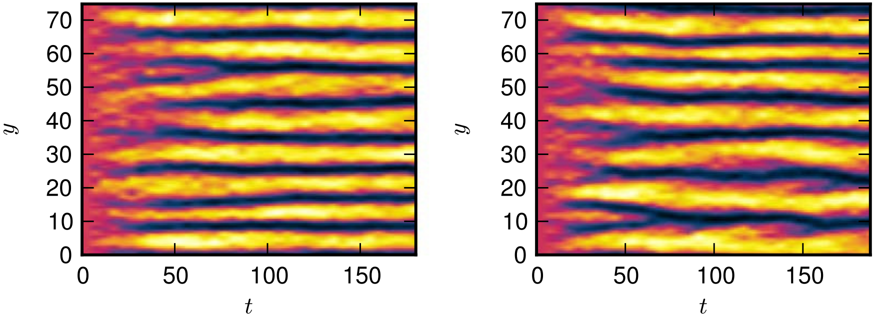


[1] M. J. Schmeits, H. A. Dijkstra, *J. Phys. Ocean.* **31**, 3435 (2001).

Bi-Stability in fluid dynamics

Multistability of atmosphere jets

Multiple attractors for atmospheric jet configurations[2]



Identical flow parameters, different forcing realizations
$\Rightarrow$ different stable zonal jet configurations[3].

[2] J. B. Parker, J. A. Krommes, *New Journal of Physics* **16**, 035006 (2014).

[3] B. F. Farrell, P. J. Ioannou, *J. Atmos. Sci.* **60**, 2101 (2003), B. F. Farrell, P. J. Ioannou, *J. Atmos. Sci.* **64**, 3652 (2007).

Extreme events in fluids

Rogue waves

Rogue waves: Probability density function unknown (but more probable than Gaussian)

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Millennium Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven *Prize Problems*. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solution to these problems, with $1 million allocated to each. During the Millennium Meeting held on May 24, 2000 at the Collège de France, Timothy Gowers presented a lecture entitled *The Importance of Mathematics*, aimed for the general public, while John Tate and Michael Atiyah spoke on the problems. The CMI invited specialists to formulate each problem.

One hundred years earlier, on August 8, 1900, David Hilbert delivered his famous lecture about open mathematical problems at the second International Congress of Mathematicians in Paris. This influenced our decision to announce the millennium problems as the central theme of a Paris meeting.

The rules for the award of the prize have the endorsement of the CMI Scientific Advisory Board and the approval of the Directors. The members of these boards have the responsibility to preserve the nature, the integrity, and the spirit of this prize.

Paris, May 24, 2000

Please send inquiries regarding the Millennium Prize Problems to prize.problems@claymath.org.

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

- Rules
- Millennium Meeting Videos

Consider the S(P)DE

$$dX_t = b(X_t)dt + \sigma dW_t$$

with

- X_t random process with $t \in [-T, 0]$, finite or infinite dimensional
- $b(X_t)$ drift term (possibly non-gradient, possibly non-linear)
- Wiener process dW_t with diffusion matrix $a = \sigma\sigma^\dagger$.

Large deviations theory:[4]

Let $X^\epsilon(t)$, $t \in [-T, 0]$ be a family of random processes, where the forcing vanishes with $\epsilon \to 0$ according to $\sigma = \sqrt{\epsilon}$. Then

$$\mathcal{P}\left\{X^\epsilon(0) \in B\right\} \asymp \exp\left(-\frac{1}{\epsilon}\inf_\psi \mathcal{I}_T[\psi]\right)$$

for the **rate function** $\mathcal{I}_T[\psi]$.

[4] A. Dembo, O. Zeitouni, *Large deviations techniques and applications*, (Springer-Verlag, Berlin, 2010).

Freidlin-Wentzell theory

Large deviations for SDEs

For the SDE above,

$$\mathcal{I}_T[X] = \frac{1}{2}\int_{-T}^{0} \mathcal{L}(X,\dot{X})\,dt, \qquad \mathcal{L}(X,\dot{X}) = \langle \dot{X} - b, a^{-1}(\dot{X} - b)\rangle$$

termed *Freidlin-Wentzell action functional*[5].

Find the minimum action e.g. via Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = a^{-1}(\dot{X} - b) \equiv P$$

$$\frac{\partial \mathcal{L}}{\partial X} = (\nabla b)^T a^{-1}(\dot{X} - b) = -(\nabla b)^T P$$

yields Hamilton's equations of motion

$$\dot{X} = aP + b$$
$$\dot{P} = -(\nabla b)^T P$$

The minimizer $(\tilde{X}, \tilde{P})$ with $\delta \mathcal{I}_T[X] = 0$ is called the **instanton**.

[5] M. I. Freidlin, A. D. Wentzell, *Random perturbations of dynamical systems*, (Springer, 1998).

Boundary conditions

Transition probabilities versus final time observable

Initial and final state of the trajectory are known:

$$X(-T) = X_{\text{start}}$$
$$X(0) = X_{\text{end}}$$

(e.g. Bi-stability, reaction paths, phase transitions)

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But: we want to measure some observable $\mathcal{O}[X] = \delta(F[X(0)] - a)$

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Modifies the equations of motion

$$\dot{X} = aP + b$$
$$\dot{P} = -(\nabla b)^T P + \lambda(\nabla F[X])\delta(t)$$

i.e. observable at final time $\Rightarrow$ **final condition** for the momentum P!

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Solving these equations gives us

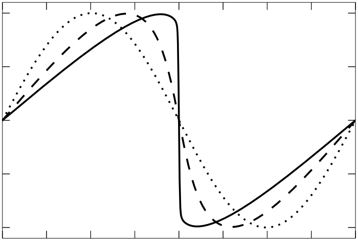
- Complete **final configuration**, fulfilling the given constraint
- Most probable **evolution** in time from initial state into this final configuration
- Corresponding **optimal force** (computable from auxiliary field P)
- Tail scaling behavior of the **PDF** of our observable

Burgers Turbulence

Burgers turbulence is considered a simple model of natural turbulence

$$u_t + uu_x - \nu u_{xx} = \eta$$

- Turbulent fields consist of smooth regions and shocks
- Exhibits strong intermittency
- Velocity gradient statistics are very skewed



Instantons have been applied to explore turbulent Burgers statistics.[6,7,8]

Goal: Use above method to analyze typical evolution of strong shocks (and deduce scaling of velocity gradient PDF tails)

[6] V. Gurarie, A. Migdal, *Phys. Rev. E* **54**, 4908 (1996).
[7] E. Balkovsky, G. Falkovich, I. Kolokolov, V. Lebedev, *Phys. Rev. Lett.* **78**, 1452 (1997).
[8] A. I. Chernykh, M. G. Stepanov, *Phys. Rev. E* **64**, 026306 (2001).

Burgers shocks

Instantons for Burgers turbulence

Application of Instanton formalism to Burgers turbulence[9]

Evolution of Burgers shocks:

$$b(u) = -uu_x + \nu u_{xx}$$
$$F[u] = u_x(0,0)$$

[9] T. Grafke, R. Grauer, T. Schäfer, *J. Phys. A* **46**, 62002 (2013).

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Question: What is the most likely evolution from $u(x)=0$ at $t=-\infty$, such that at the end (i.e. $t=0$) we have a high gradient in the origin $u_x(x=0,t=0)=z$ (shock)?

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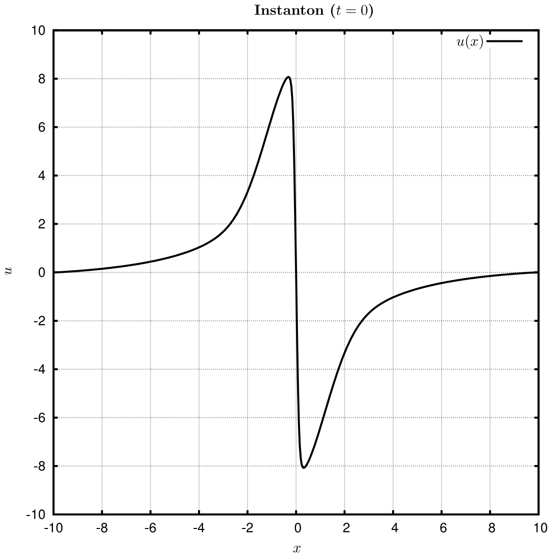

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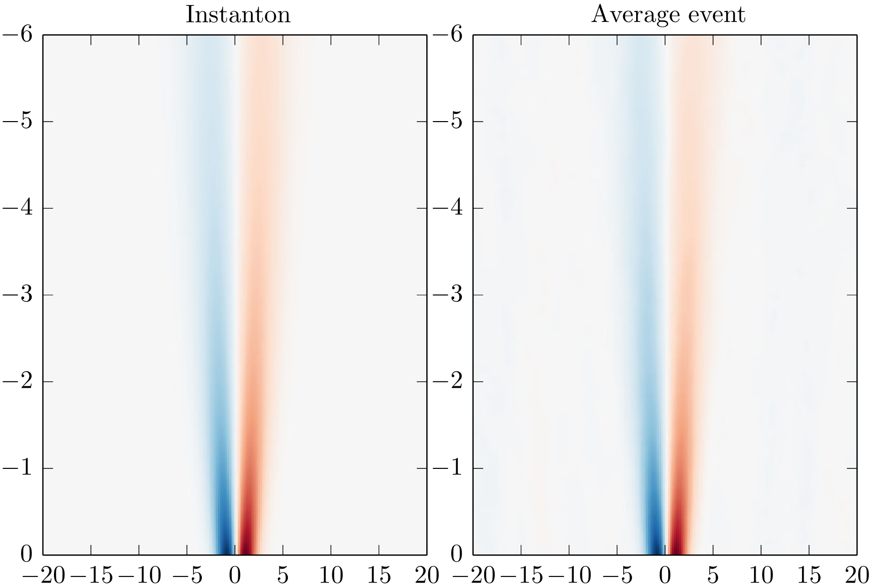
Burgers turbulence

Extreme events versus instantons



PDF tail scaling

Instanton predictions for the probability of rare events

From large deviations we know:

$$\mathcal{P}\{F[X(t=0)] \in B\} \sim \exp(-\mathcal{I}[X_{\text{inst}}])$$

Knowledge of the **instanton** implies knowledge of the **PDF** tails.

[10] V. Gurarie, A. Migdal, *Phys. Rev. E* **54**, 4908 (1996).
[11] E. Balkovsky, G. Falkovich, I. Kolokolov, V. Lebedev, *Phys. Rev. Lett.* **78**, 1452 (1997).
[12] T. Gotoh, *Phys. Fluids* **11**, 2143–2148 (1999).

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Assuming a specific form of the PDF for velocity gradient $u_x = z$:

$$\mathcal{P}(z) \sim \exp(-|z|^{\vartheta}) \qquad \Rightarrow \qquad \mathcal{I}[X_{\text{inst}}] \sim |z|^{\vartheta}$$

yields[10,11]:

Left tail (shocks):

$$\lim_{u_x \to -\infty} \vartheta = \frac{3}{2}$$

Right tail (rarefaction waves):

$$\lim_{u_x \to \infty} \vartheta = 3$$

[10] V. Gurarie, A. Migdal, *Phys. Rev. E* **54**, 4908 (1996).
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But: measured left tail more like[12] $\vartheta = 1.15 \neq \frac{3}{2}$

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[11] E. Balkovsky, G. Falkovich, I. Kolokolov, V. Lebedev, *Phys. Rev. Lett.* **78**, 1452 (1997).
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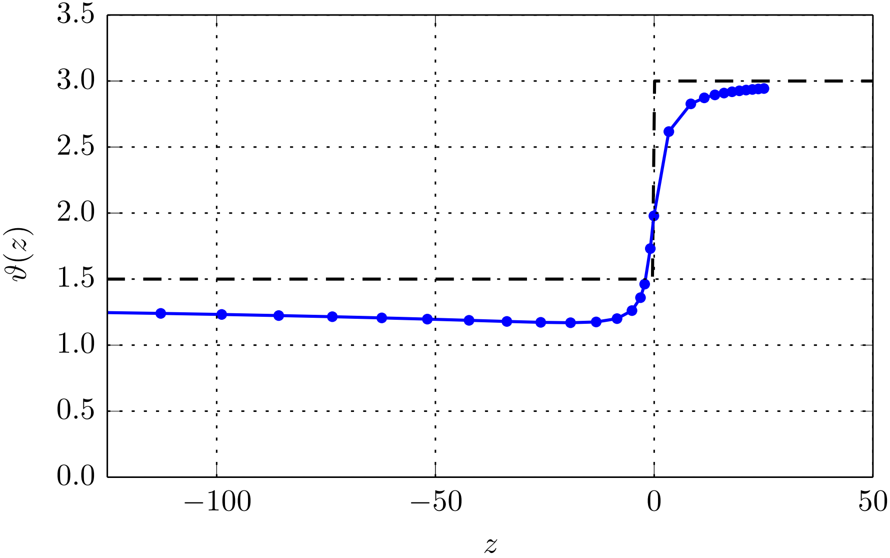
PDF tail scaling

Burgers instantons versus DNS



Limiting case, Instanton

PDF tail scaling

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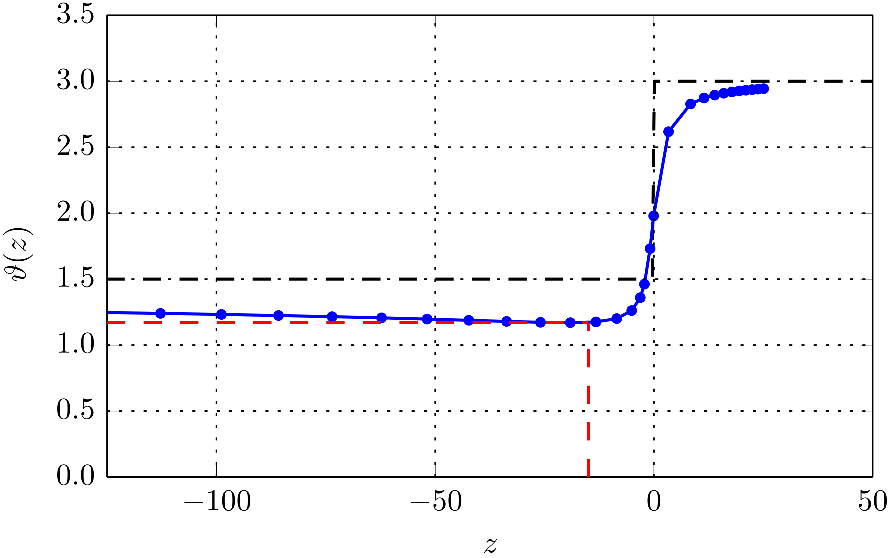

Limiting case, Instanton, Gotoh DNS

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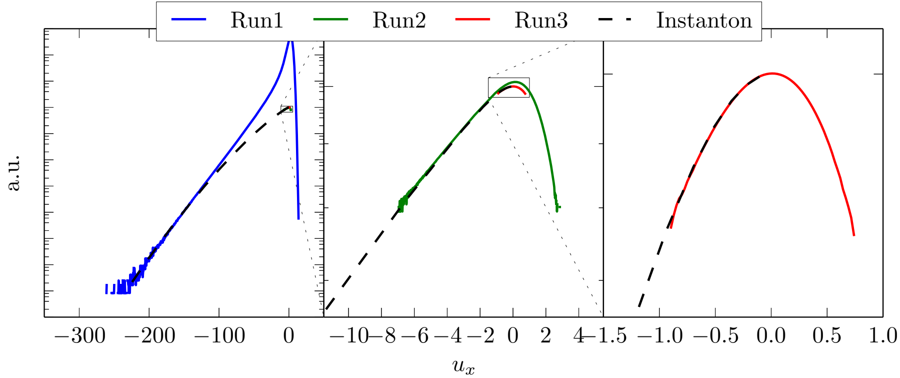
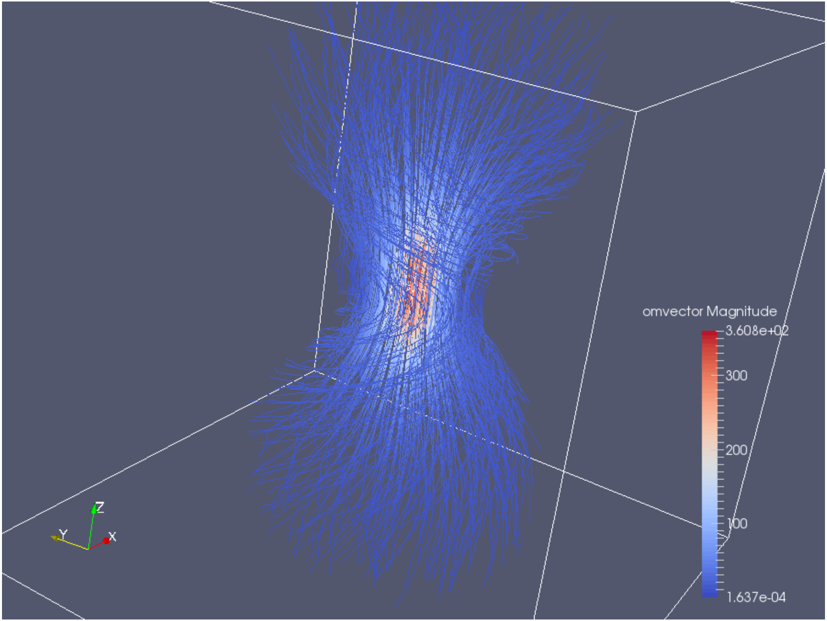


from T. Grafke, R. Grauer, T. Schäfer, E. Vanden-Eijnden, *EPL* **109**, 34003 (2015)

Instanton for the 3D Navier-Stokes equations



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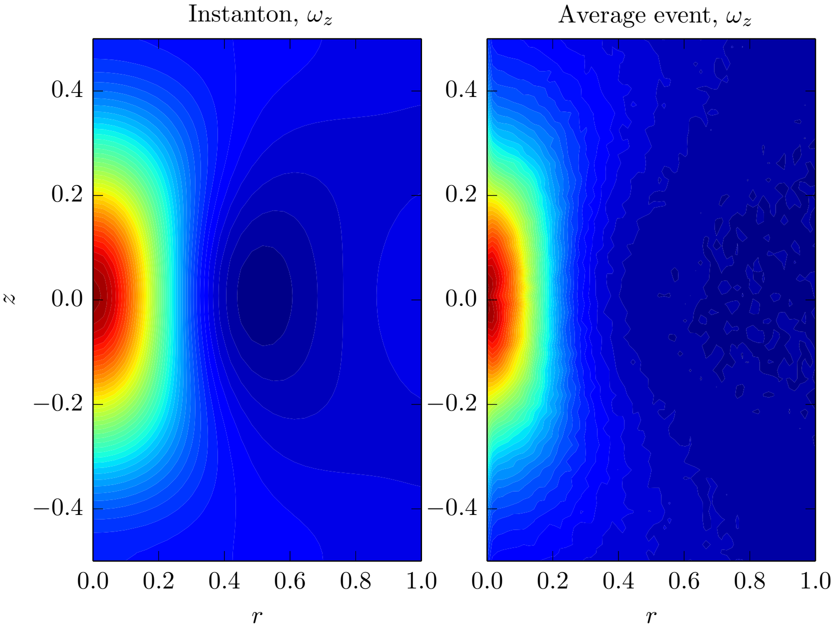

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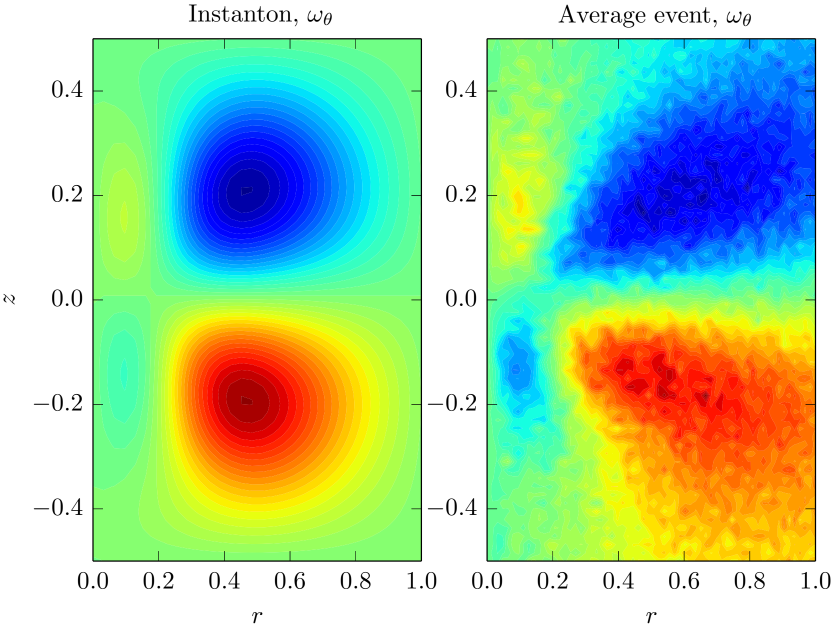

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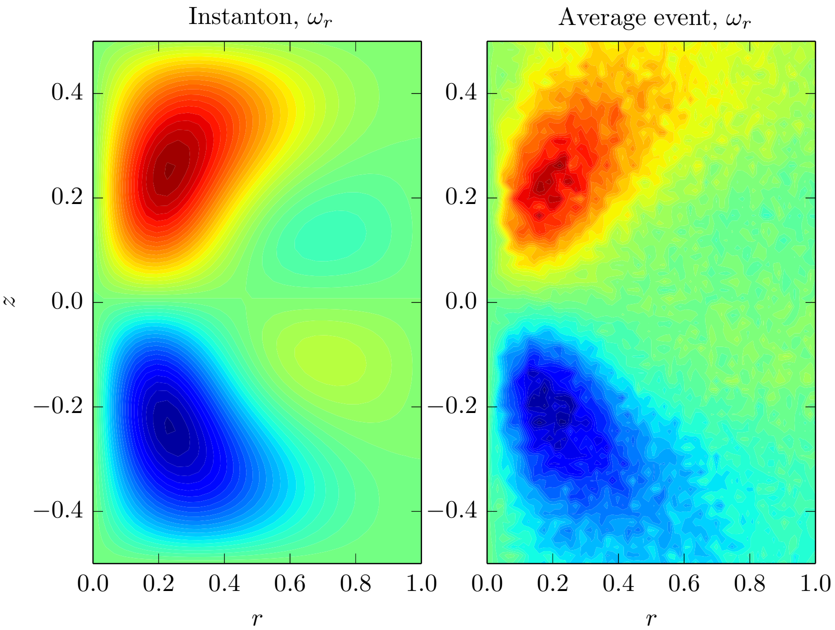

Conclusion

In general

- we are able to compute the **instanton** for the evolution from field at rest to arbitrary final states.
- we can use the instanton configuration to make **quantitative predictions** about the statistics of **rare events**.

For Burgers equation:

- We can **recover** the instanton configuration in stochastic Burgers flows.
- We can **exlain the discrepancy** between measurements in DNS and analytical estimates from the instanton approach.
- We can **predict the PDF** for a wide range of Reynolds numbers.

For other equations:

- Similar treatment of actual turbulence in **2D** or **3D** is in reach.

Numerical Computation of Instanton configuration

We want to solve the instanton equations numerically! Problems:

- Starting from a stable fixed point of the deterministic dynamics:

  $$T \to \infty$$

  How to discretize?

  Solution: Minimize on space of arc-length parametrized curves, $\|\dot{x}\|_a = 1$, (*geometric* instanton[13,14])

- Fluid dynamics (esp. Turbulence): Large number of degrees of freedom.

  E.g. 2D fluid, space resolution 1024×1024, number of timesteps $\approx 10^4$

  $$\Longrightarrow \quad N \approx 10^{10} \quad \text{(!)}$$

  Solution: Various optimizations (Equations of motion, Multigrid-like recursive time integration, compact support of correlation for large scale forcing, GPUs)[15]

[13] M. Heymann, E. Vanden-Eijnden, *Commun. Pure Appl. Math.* **61**, 1053 (2008).
[14] T. Grafke, R. Grauer, T. Schäfer, E. Vanden-Eijnden, *Multiscale Modeling & Simulation* **12**, 566–580 (2014).
[15] T. Grafke, R. Grauer, S. Schindel, *arXiv:1410.6331 (physics.flu-dyn)*, arXiv: 1410.6331 (Oct. 2014).