A geometric framework for interpreting and parameterising geostrophic ocean eddies

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1° (climate) resolution

1/12° resolution

Re ~ 1

Re ~ 10³

(MICOM, University of Miami)

Classical paradigm for location/structure of ocean eddies:

Eady (1949) model of baroclinic instability

- uniform rotation
- uniform stratification
- uniform shear
- opposing potential vorticity gradients at upper and lower boundaries

Most unstable mode:

- uniform shear
- leans against mean shear

Energy growth rate for most unstable mode:

\[ 0.61 \frac{f_0}{N_0} \frac{\partial u}{\partial z} \sim 0.3 \text{ day}^{-1} \text{ \cdot atmosphere} \]

\[ 0.03 \text{ day}^{-1} \text{ \cdot ocean} \]

Length scale of instability characterised by Rossby deformation radius:

\[ L_d = \frac{N_0 H}{f_0} \sim 1000 \text{ km} \text{ \cdot atmosphere} \]

\[ 50 \text{ km} \text{ \cdot ocean} \]
Gent and McWilliams (1990):

**adiabatic parameterisation of baroclinic instability**

eddies mix along isopycnals (Redi 1982) ...

... and advect by an *eddy bolus velocity* - flattens isopycnals (Gent et al. 1995)

\[
\mathbf{u}^* = \frac{\partial}{\partial z} \left( \kappa \frac{\nabla b}{\partial b/\partial z} \right), \quad w^* = -\nabla \cdot \left( \kappa \frac{\nabla b}{\partial b/\partial z} \right),
\]

removes available potential energy

can relate eddy diffusivity, \( \kappa \), to mean flow (e.g., Visbeck et al. 1997)
or eddy energy (Eden and Greatbatch 2008)
Alternative paradigm: potential vorticity mixing

often advocated ... rarely implemented in ocean GCMs!

Idea: potential vorticity  \( q = \frac{f + \xi}{h} \) is materially conserved in absence of forcing/dissipation:

\[
\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0
\]

stirred and mixed along isopycnals \( \Rightarrow \) down-gradient closure, \( q' u' = -\kappa \nabla \rho \bar{q} \) ?

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PV mixing problem 1: conservation of energy

\[ e.g. \text{, freely-decaying turbulence over a seamount} \quad (\text{Adcock and Marshall, 2000}) \]
PV mixing problem 2: conservation of momentum

periodic channel:

\[
\begin{array}{c}
\text{eddy force} \\
\int \int \int q' v' \, dx \, dy \, dz = 0 \\
\text{conservation of momentum}
\end{array}
\]

not satisfied by down-gradient potential vorticity closures without additional constraints

(Green, 1970; J. Marshall, 1981)

e.g., here \( q' v' = -\kappa \partial q / \partial y \) only consistent if \( \kappa = 0 \)

note: this is the Charney-Stern stability condition

Take-home message:

Eddies mix potential vorticity along density surfaces ...
... subject to constraints of energy and momentum conservation

Goal of this work:

Develop framework for interpreting and parameterising eddy potential vorticity fluxes in which the relevant symmetries and conservation laws are preserved.

Work with quasi-geostrophic “residual-mean” equations:

\[
\frac{\partial \overline{u}_g}{\partial t} + \ldots = -\kappa \times \overline{q' u'}
\]

eddy force

how to parameterise?

(Maddison and Marshall, 2013; cf. Young, 2012)
Key idea:

Write \textit{eddy potential vorticity flux} (or eddy force) as \textit{divergence of an eddy stress tensor}:

\[
\overline{q' \mathbf{u}'} = \nabla \cdot \left( \begin{array}{ccc} -N & M - P & 0 \\ M - P & N & 0 \\ R & S & 0 \end{array} \right)
\]

(Plumb 1986)

“Taylor identity”

where: \[M = \frac{\overline{v'^2} - \overline{u'^2}}{2}\quad N = -\overline{u'v'}\quad \text{Reynolds stresses}\]

\[P = \frac{\overline{b'^2}}{2N_0^2}\quad \text{eddy potential energy}\]

\[R = \frac{f_0}{N_0^2} \frac{\overline{w'b'}}{S} = \frac{f_0}{N_0^2} \frac{\overline{v'b'}}{S} \quad \text{eddy buoyancy flux / “eddy form stress”}\]

Why do this?!!!

1. This is a mathematical identity! (down-gradient flux ≠ divergence of a tensor)

2. Momentum constraints preserved with appropriate boundary conditions:

\[
\frac{\partial \overline{\mathbf{u}g}}{\partial t} + \cdots = \nabla \cdot (\text{eddy momentum fluxes})
\]

3. Reduces to Gent and McWilliams (1990) / Greatbatch and Lamb (1990) if we parameterise only the vertical momentum fluxes.

Therefore a natural framework for extending Gent and McWilliams.
4. Suppose we solve an eddy energy equation (Eden and Greatbatch, 2008):

$$\frac{\partial E}{\partial t} + \cdots = -\overline{u}_g \cdot \nabla \text{(eddy force)} = \overline{u}_g \cdot \mathbf{k} \times \overline{q}^T \overline{u}$$

This eddy energy gives a bound on the magnitude of the eddy stress tensor:

$$\frac{1}{2} \left[ (-N)^2 + (M - P)^2 + (M + P)^2 + N^2 + \frac{N_0^2}{f_0^2} (R^2 + S^2) \right] \leq E^2$$

This means there are no remaining dimensional unknowns!

5. This allows us to rewrite the eddy stress tensor in terms of the eddy energy, two non-dimensional eddy anisotropies, and three eddy angles:

$$M = -\gamma_m E \cos 2\phi_m \cos^2 \lambda \quad N = \gamma_m E \sin 2\phi_m \cos^2 \lambda \quad P = E \sin^2 \lambda$$

$$R = \gamma_b \frac{f_0}{N_0} E \cos \phi_b \sin 2\lambda \quad S = \gamma_b \frac{f_0}{N_0} E \sin \phi_b \sin 2\lambda$$

horizontal angles vertical angle

e.g., barotropic eddies: (plan view) $$\gamma_m = 0 \quad \overline{u'}u' = 0$$

$$\gamma_m \rightarrow 1 \quad \overline{u'}u' > 0$$

“wave-like”
Why do this?!!!

Equivalent to **eddy variance ellipses** used with altimetric / float data:

\[
\frac{q' u'}{N} = \nabla \cdot \begin{pmatrix} -N & M - P & 0 \\ M - P & N & 0 \\ R & S & 0 \end{pmatrix}
\]

(Chelton and Schlax 2003)

(Morrow et al. 1994)

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6. Eddy angles have a strong connection with classical stability theory:

- Eddies lean “against” mean shear => extract energy from mean flow - **instability**;
- Eddies lean “into” mean shear => return energy to mean flow - **stability**.

(Waterman et al. 2011)
Application: Eady model

most unstable mode:

Eady energy budget:

\[
\frac{\partial}{\partial t} \iiint E \, dx \, dy \, dz = - \iiint \mathbf{u} \cdot \mathbf{q} \, dx \, dy \, dz
\]

\[
= - \iiint \mathbf{u} \frac{\partial S}{\partial z} \, dx \, dy \, dz
\]

\[
= \iiint \frac{\partial \mathbf{u}}{\partial z} S \, dx \, dy \, dz
\]

\[
= \frac{\alpha f_0}{N_0} \frac{\partial \mathbf{u}}{\partial z} \int \iiint E \, dx \, dy \, dz
\]

\[Eady \, growth \, rate:\]

\[-1 \leq \alpha \leq 1 \quad \text{if} \quad \alpha = 0.61\]

How anisotropic are the eddies?

3-layer QG model

wind stress

\[
(a) \; H_1 \psi_1 \ (Sv)
\]

\[
(b) \; H_2 \psi_2 \ (Sv)
\]

\[
(c) \; H_3 \psi_3 \ (Sv)
\]

\[
(d) \; q_1 \ (10^{-4} \, s^{-1})
\]

\[
(e) \; q_2 \ (10^{-4} \, s^{-1})
\]

\[
(f) \; q_3 \ (10^{-4} \, s^{-1})
\]
eddy anisotropies

layer 1                         layer 2                        layer 3

buoyancy

momentum

\[ N = \gamma_m E \sin 2\phi_m \cos^2 \lambda \]
\[ S = \gamma_b \frac{f_0}{N_0} E \sin \phi_b \sin 2\lambda \]

What sets the eddy angles?  
- for linear Rossby waves: refraction

(e.g., Buhler and McIntyre, 2005)
For a barotropic jet on a beta plane, a sharp change in the eddy-mean flow interaction occurs. The evolution of the jet is described using linear waves, the eddy ellipse still possesses similar geometric properties.

In terms of CRWs perspective, all the vorticity waves become in phase (e.g. as in Fig. (3.2a)), hence the eddy ellipse tilt is almost constant within the layers. Gradually, the eddy orientation becomes slanted once the jet stabilizes, which have a clear signature in the geometric description of the eddies. During the initial development, anisotropy is localized mainly at the three interfaces of the jet, and the unstable NM is constant and positive (negative) in the cyclonic (anti-cyclonic) side of the jet, fluxing the mean PV gradient

Compared are actual tilt (red), estimated value (dashes black), ray tracing results from numerical (blue) and analytic (green stars) solutions. See text for explanation.

Figure 4.2: Eddy ellipse tilt for (a) inward radiating ray (solid blue) and analytic (green stars) solutions. See text for explanation.
Mixing of potential vorticity?

If we: (i) solve an eddy potential enstrophy ($\bar{q}^2$) budget; (ii) include dissipation of $\bar{q}^2$ ($=\text{potential vorticity mixing}$); (ii) ensure $\bar{q}'\overline{u'}$ vanishes when $\bar{q}^2$ vanishes; [use another bound on divergence of eddy stress tensor?]

then Arnold’s first stability theorem is preserved.

Physical interpretation? (Marshall and Adcroft, 2010)

Eddy energy equation:
\[
\frac{\partial}{\partial t} \frac{u' \cdot u'}{2} + \nabla \cdot (\ldots) = +\bar{q}'u' \cdot \nabla \bar{q}
\]

Eddy enstrophy equation:
\[
\frac{\partial}{\partial t} \frac{\bar{q}^2}{2} + \nabla \cdot (\ldots) = -\bar{q}'u' \cdot \nabla \bar{q}
\]

If $d\bar{q}/d\bar{q} > 0$, eddy energy can grow only at the expense of eddy potential enstrophy.

$\Rightarrow$ stable (in the sense of Lyapunov) - Arnold’s first stability theorem.

Coordinate-invariant derivation (Maddison and Marshall, 2013)

quasigeostrophic PV equation:
\[
\partial_t \bar{q} + \left( [u_t]^a \right)_{,a} \bar{q}_{,b} = -T^{ab}_{,ab} \]

double divergence $\Rightarrow$ 2 forms of gauge freedom

eddy flux tensor
\[
g_{ac}T^{cb} = \begin{pmatrix} N & M - K & R \\ M + K & -N & S \\ 0 & 0 & 0 \end{pmatrix}
\]

“residual-mean”
\[
\begin{pmatrix} N & M + K & 0 \\ M - K & -N & 0 \\ R & S & 0 \end{pmatrix}
\]

Cronin (1996)
\[
\begin{pmatrix} N & M - K & 0 \\ M + K & -N & 0 \\ R & S & 0 \end{pmatrix}
\]

“half-residual mean”
\[
\begin{pmatrix} N & M + \frac{1}{2}R & \frac{1}{2}S \\ M - N & \frac{1}{2}S & 0 \\ \frac{1}{2}R & \frac{1}{2}S & 0 \end{pmatrix}
\]

Plumb (1986)
\[
\begin{pmatrix} N & M - P & 0 \\ M + P & -N & 0 \\ R & S & 0 \end{pmatrix}
\]

Hoskins et al. (1983) “E-vector”
\[
\begin{pmatrix} N & 2M & 0 \\ 0 & -N & 0 \\ R & S & 0 \end{pmatrix}
\]

Approach generalises to isopycnal thickness-weighted primitive equations
(cf. Young, 2012)
Summary of key points

• Geostrophic eddies are fundamental in setting the structure and circulation of the ocean.

• Preserving symmetries and conservation laws in models with parameterised eddies
  ⇒ classical stability conditions carry over.

• Down-gradient eddy potential vorticity flux closures are inconsistent with the underlying
  mathematical structure of the eddy-mean flow interaction.

• Gent and McWilliams is consistent with this underlying mathematical structure.

• New geometric framework for diagnosing and interpreting eddy-mean flow interactions.

• Much left to do, e.g.:

  - simple extension of Gent and McWilliams to include up-gradient momentum fluxes;
  - adjoint methods to optimise choice of parameters (Julian Mak);
  - eddy-topography interactions;
  - diagnostics of eddy-mean flow interactions in Southern Ocean (Andreas Klocker);
  - applications to planetary atmospheres?
  - ...

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