A geometric framework for interpreting and parameterising geostrophic ocean eddies

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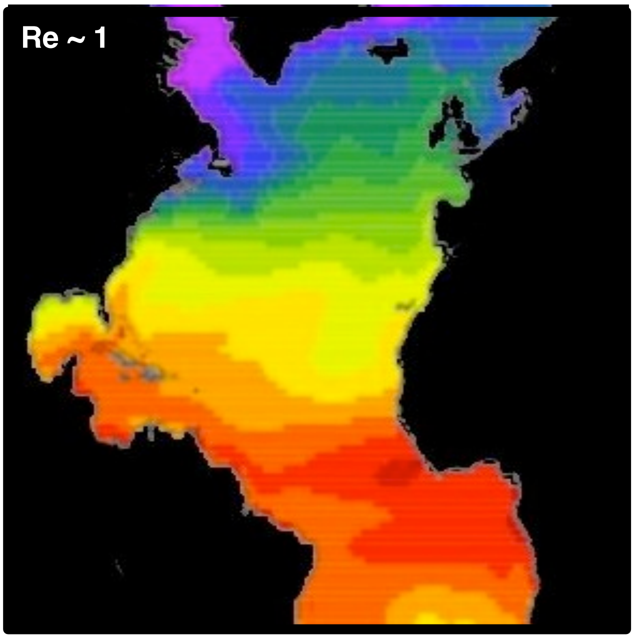
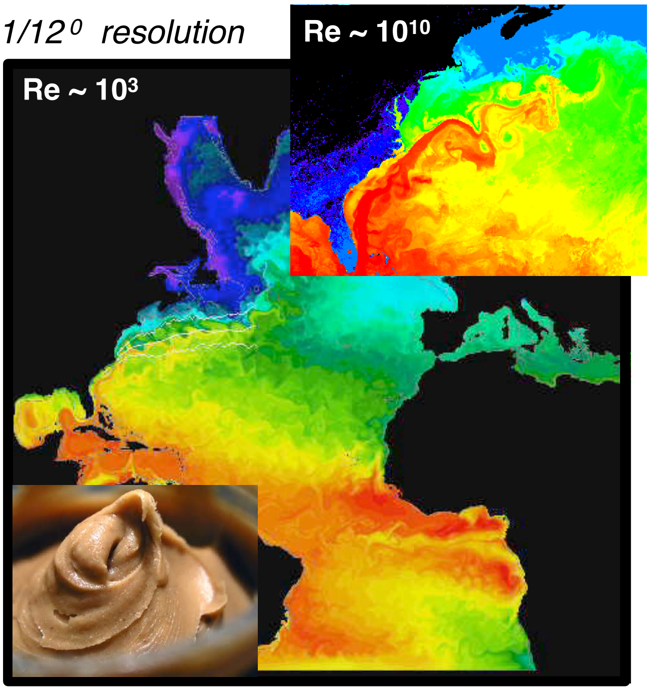
1^0 (climate) resolution



$1/12^0$ resolution



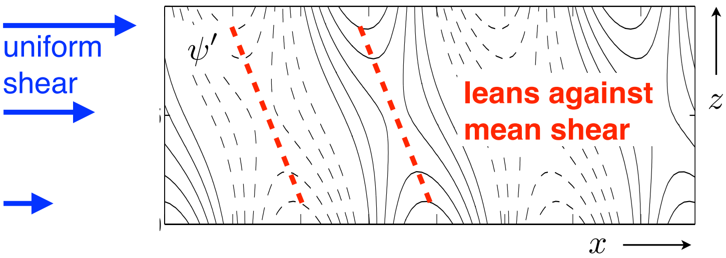
(MICOM, University of Miami)

Classical paradigm for location/structure of ocean eddies:

Eady (1949) model of baroclinic instability

- uniform rotation
- uniform stratification
- uniform shear
- opposing potential vorticity gradients at upper and lower boundaries

most **unstable** mode:



(figure: adapted from Vallis 2006)

energy growth rate for most unstable mode:

$$0.61 \frac{f_0}{N_0} \frac{\partial u}{\partial z} \sim$$

0.3 day^{-1} - atmosphere

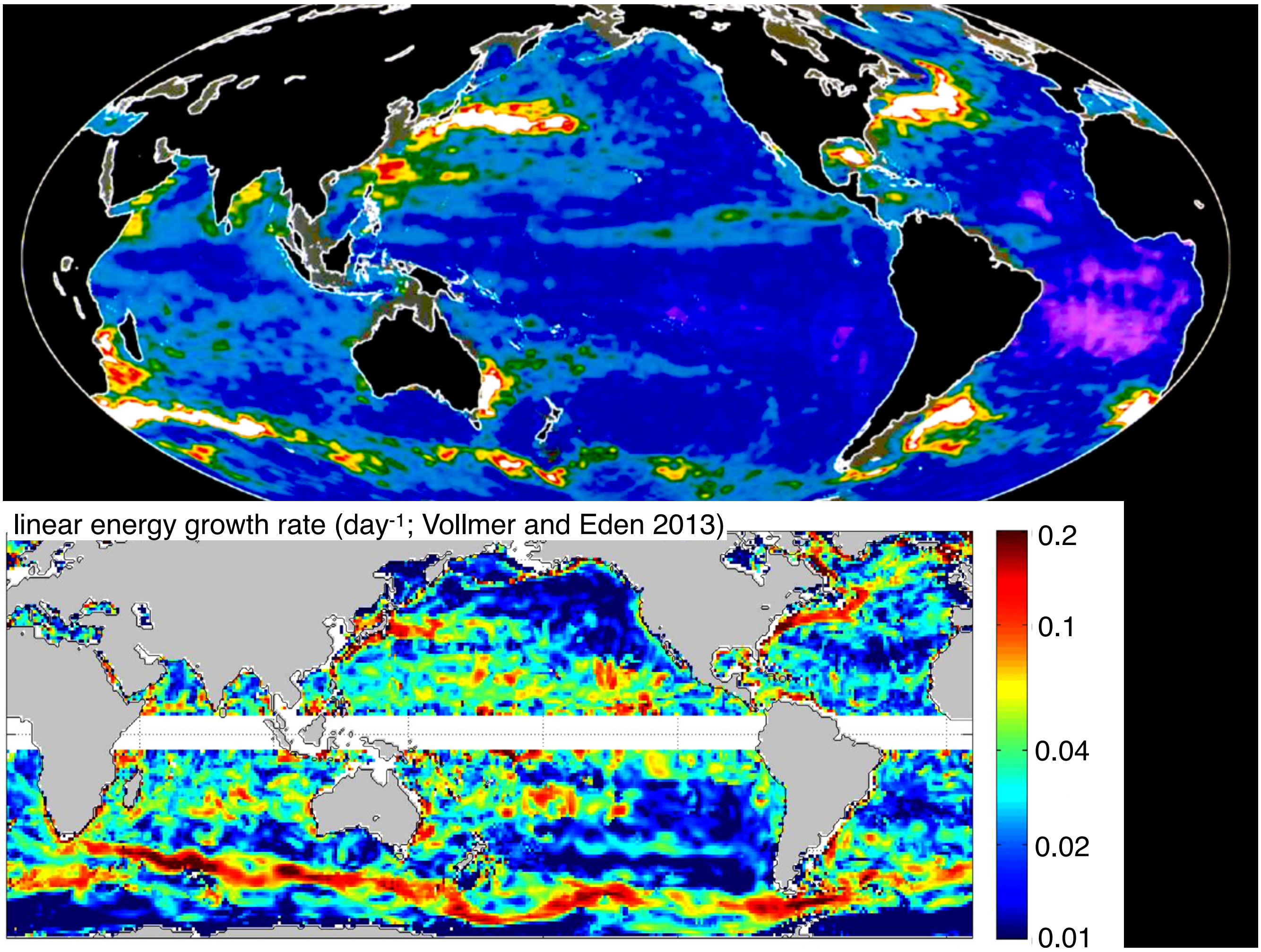
0.03 day^{-1} - ocean

length scale of instability characterised by
Rossby deformation radius:

$$L_d = \frac{N_0 H}{f_0} \sim$$

1000 km - atmosphere

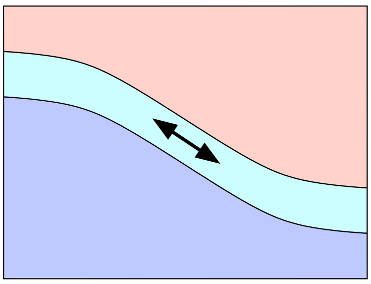
50 km - ocean



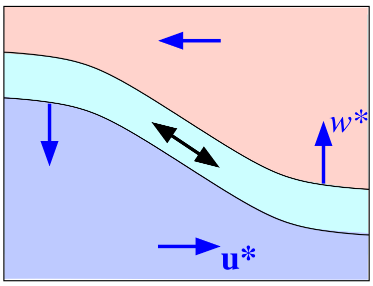
Gent and McWilliams (1990):

adiabatic parameterisation of baroclinic instability

eddies mix along isopycnals (Redi 1982) ...



... and advect by an *eddy bolus velocity* - flattens isopycnals (Gent et al. 1995)



$$\mathbf{u}^* = \frac{\partial}{\partial z}\left(\kappa \frac{\nabla b}{\partial b/\partial z}\right), \quad w^* = -\nabla \cdot \left(\kappa \frac{\nabla b}{\partial b/\partial z}\right),$$

removes available potential energy

can relate eddy diffusivity, κ, to mean flow (e.g., Visbeck et al. 1997)
or eddy energy (Eden and Greatbatch 2008)

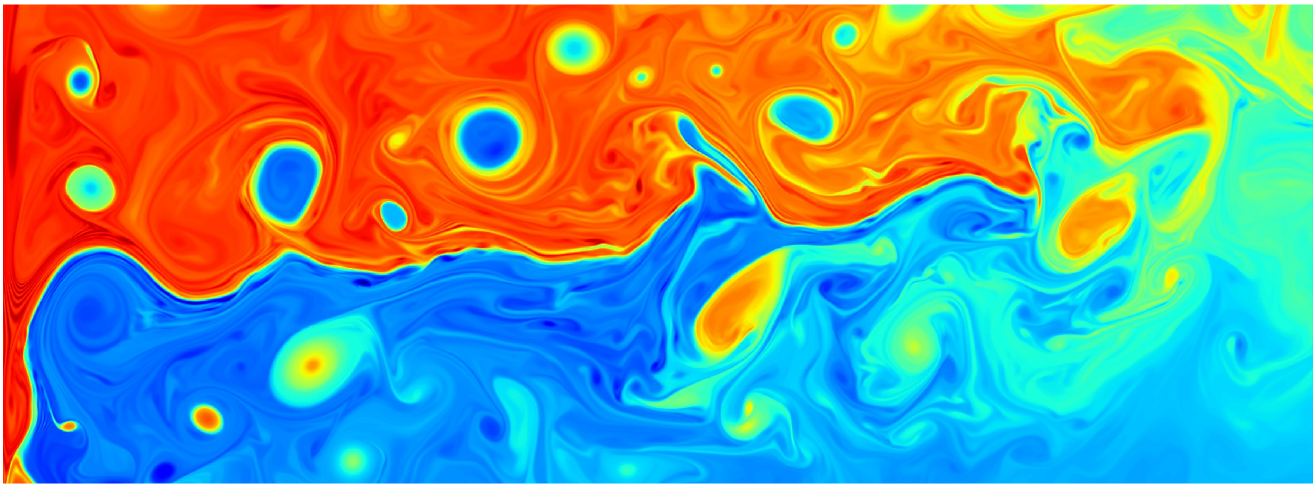
Alternative paradigm: potential vorticity mixing

often advocated ... rarely implemented in ocean GCMs!

Idea: potential vorticity $q = \dfrac{f + \xi}{h}$ is materially conserved in absence of forcing/dissipation:

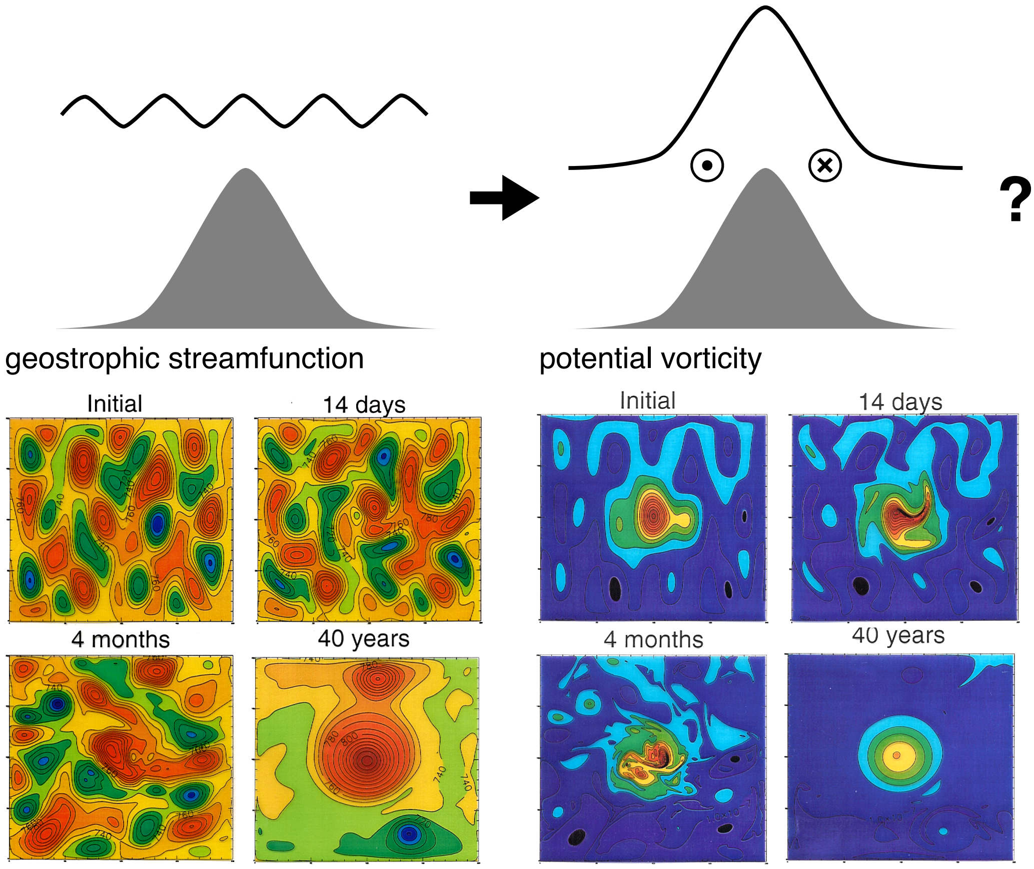
$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0$$

stirred and mixed along isopycnals ⇒ down-gradient closure, $\overline{q'\mathbf{u}'} = -\kappa \nabla_\rho \overline{q}$?



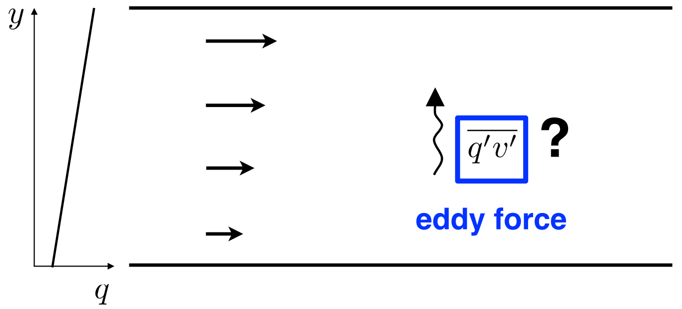
PV mixing problem 1: conservation of energy

e.g. , freely-decaying turbulence over a seamount   (Adcock and Marshall, 2000)



PV mixing problem 2: conservation of momentum

periodic channel:



$$\iiint \overline{q'v'}\, dx\, dy\, dz = 0$$

conservation of momentum

not satisfied by down-gradient potential vorticity closures without additional constraints
(Green, 1970; J. Marshall, 1981)

e.g., here $\overline{q'v'} = -\kappa\, \partial q/\partial y$ only consistent if $\kappa = 0$

note: this is the Charney-Stern stability condition

Take-home message:

Eddies mix potential vorticity along density surfaces ...
... subject to constraints of energy and momentum conservation

Goal of this work:

Develop framework for **interpreting** and **parameterising** eddy potential vorticity fluxes in which the relevant **symmetries and conservation laws are preserved**.

Work with quasi-geostrophic "residual-mean" equations:

$$\frac{\partial \overline{\mathbf{u}}_g}{\partial t} + \quad \cdots \quad = \boxed{-\mathbf{k} \times \overline{q'\mathbf{u}'}}$$

eddy force

how to parameterise?

(Maddison and Marshall, 2013; cf. Young, 2012)

Key idea:

Write **eddy potential vorticity flux** (or eddy force)
as **divergence of an eddy stress tensor**:

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

(Plumb 1986)

"Taylor identity"

where: $M = \dfrac{\overline{v'^2 - u'^2}}{2}$ $\quad N = -\overline{u'v'}$ Reynolds stresses

$$P = \frac{\overline{b'^2}}{2N_0^2}$$ eddy potential energy

$R = \dfrac{f_0}{N_0^2}\overline{u'b'}$ $\quad S = \dfrac{f_0}{N_0^2}\overline{v'b'}$ eddy buoyancy flux / "eddy form stress"

Why do this?!!!

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

1. This is a mathematical identity! (down-gradient flux ≠ divergence of a tensor)

2. Momentum constraints preserved with appropriate boundary conditions:

$$\frac{\partial \overline{\mathbf{u}}_g}{\partial t} + \quad \cdots \quad = \nabla \cdot (\text{eddy momentum fluxes})$$

3. Reduces to Gent and McWilliams (1990) / Greatbatch and Lamb (1990) if we parameterise only the vertical momentum fluxes.

Therefore a natural framework for extending Gent and McWilliams.

Why do this?!!!

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

4. Suppose we solve an **eddy energy** equation (Eden and Greatbatch, 2008):

$$\frac{\partial \overline{E}}{\partial t} + \quad \cdots \quad = -\overline{\mathbf{u}}_g \cdot \nabla(\text{eddy force}) = \overline{\mathbf{u}}_g \cdot \mathbf{k} \times \overline{q'\mathbf{u}'}$$

This **eddy energy** gives a bound on the **magnitude of the eddy stress tensor**:

$$\frac{1}{2}\left[(-N)^2 + (M-P)^2 + (M+P)^2 + N^2 + \frac{\mathcal{N}_0^2}{f_0^2}(R^2+S^2)\right] \le E^2$$

This means there are **no remaining dimensional unknowns**!

Why do this?!!!

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

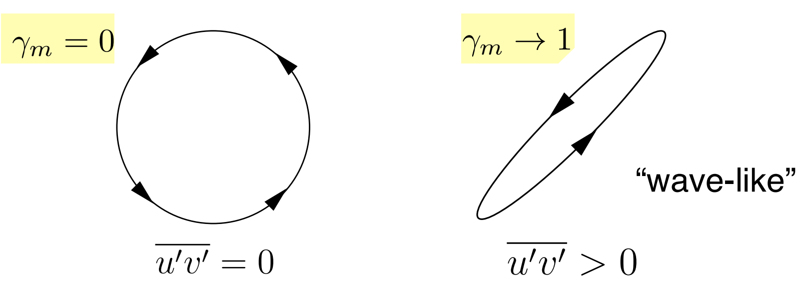
5. This allows us to rewrite the eddy stress tensor in terms of the **eddy energy**, two non-dimensional **eddy anisotropies**, and three **eddy angles**:

$$M = -\gamma_m E \cos 2\phi_m \cos^2\lambda \qquad N = \gamma_m E \sin 2\phi_m \cos^2\lambda \qquad P = E \sin^2\lambda$$

$$R = \gamma_b \frac{f_0}{\mathcal{N}_0} E \cos\phi_b \sin 2\lambda \qquad S = \gamma_b \frac{f_0}{\mathcal{N}_0} E \sin\phi_b \sin 2\lambda$$

horizontal angles   vertical angle

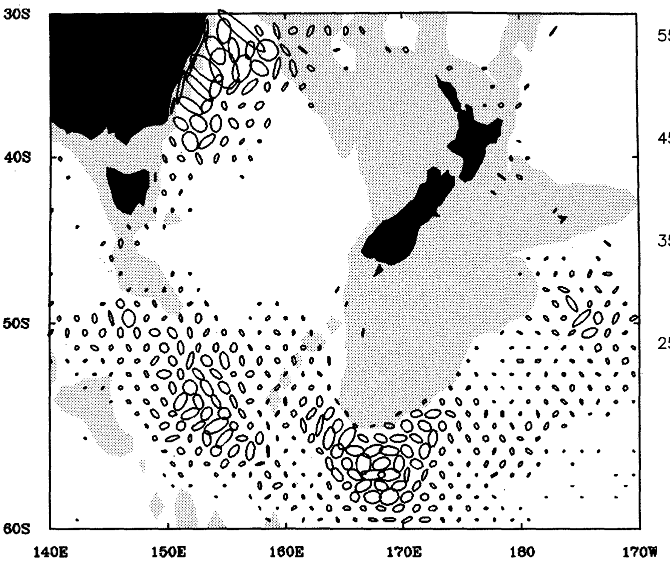
e.g., barotropic eddies:
(plan view)



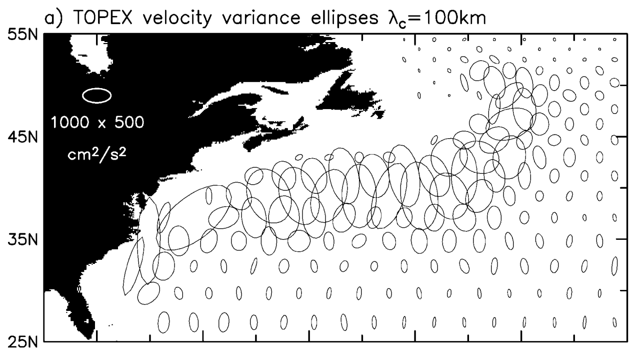
Why do this?!!!

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

Equivalent to **eddy variance ellipses** used with altimetric / float data:



(Morrow et al. 1994)



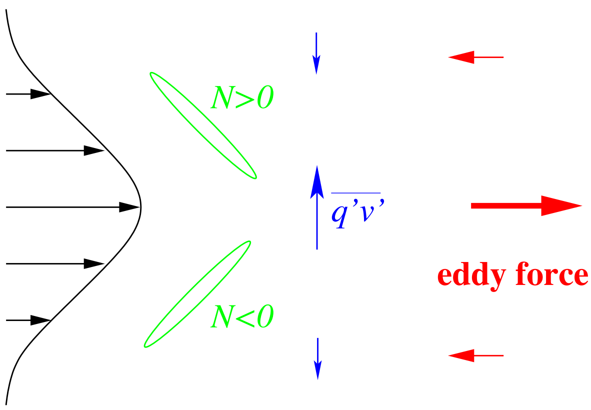
(Chelton and Schlax 2003)

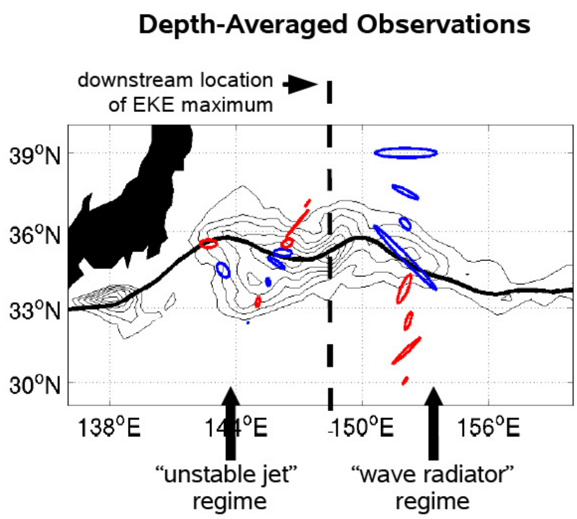
Why do this?!!!

$$\overline{q'\mathbf{u}'} = \nabla \cdot \begin{pmatrix} -N & M-P & 0 \\ M-P & N & 0 \\ R & S & 0 \end{pmatrix}$$

6. Eddy angles have a strong connection with classical stability theory:

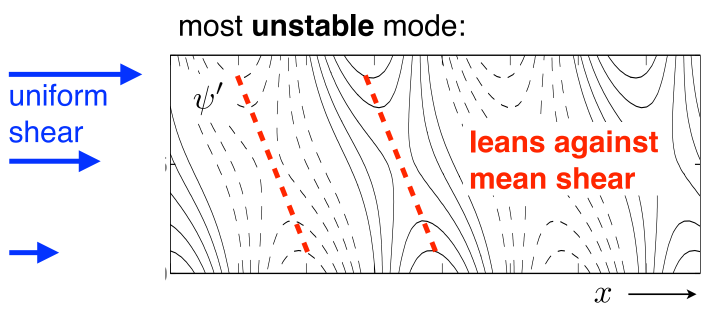
eddies lean "against" mean shear ⇒ extract energy from mean flow - **instability**;
eddies lean "into" mean shear ⇒ return energy to mean flow - **stability**.





(Waterman et al. 2011)

Application: Eady model



Eddy energy budget:

$$\frac{\partial}{\partial t}\iiint E\,dx\,dy\,dz = -\iiint \overline{u}\,\overline{q'v'}\,dx\,dy\,dz$$ **u . (eddy force)**

$$= -\iiint \overline{u}\frac{\partial S}{\partial z}\,dx\,dy\,dz$$ *substitute eddy stress tensor*

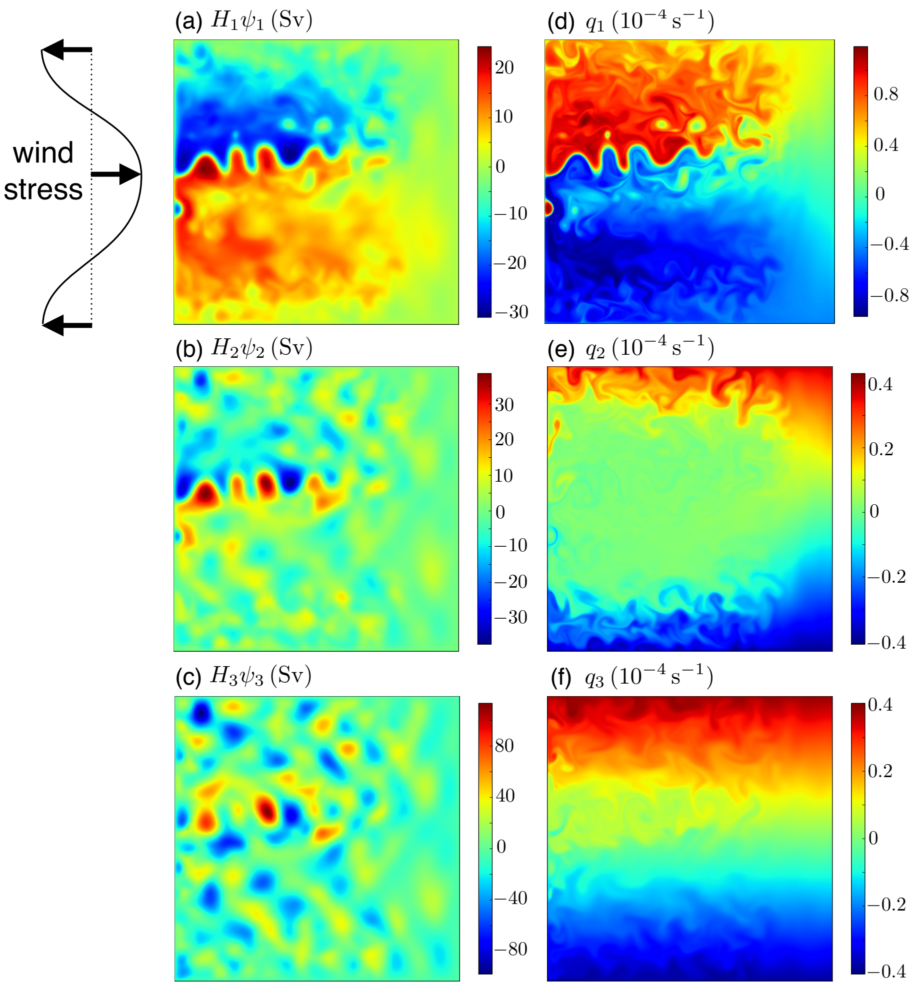
$$= \iiint \frac{\partial \overline{u}}{\partial z} S\,dx\,dy\,dz$$ *integrate by parts*

$$= \boxed{\alpha\frac{f_0}{N_0}\frac{\partial \overline{u}}{\partial z}}\iiint E\,dx\,dy\,dz$$ *apply energy bound*

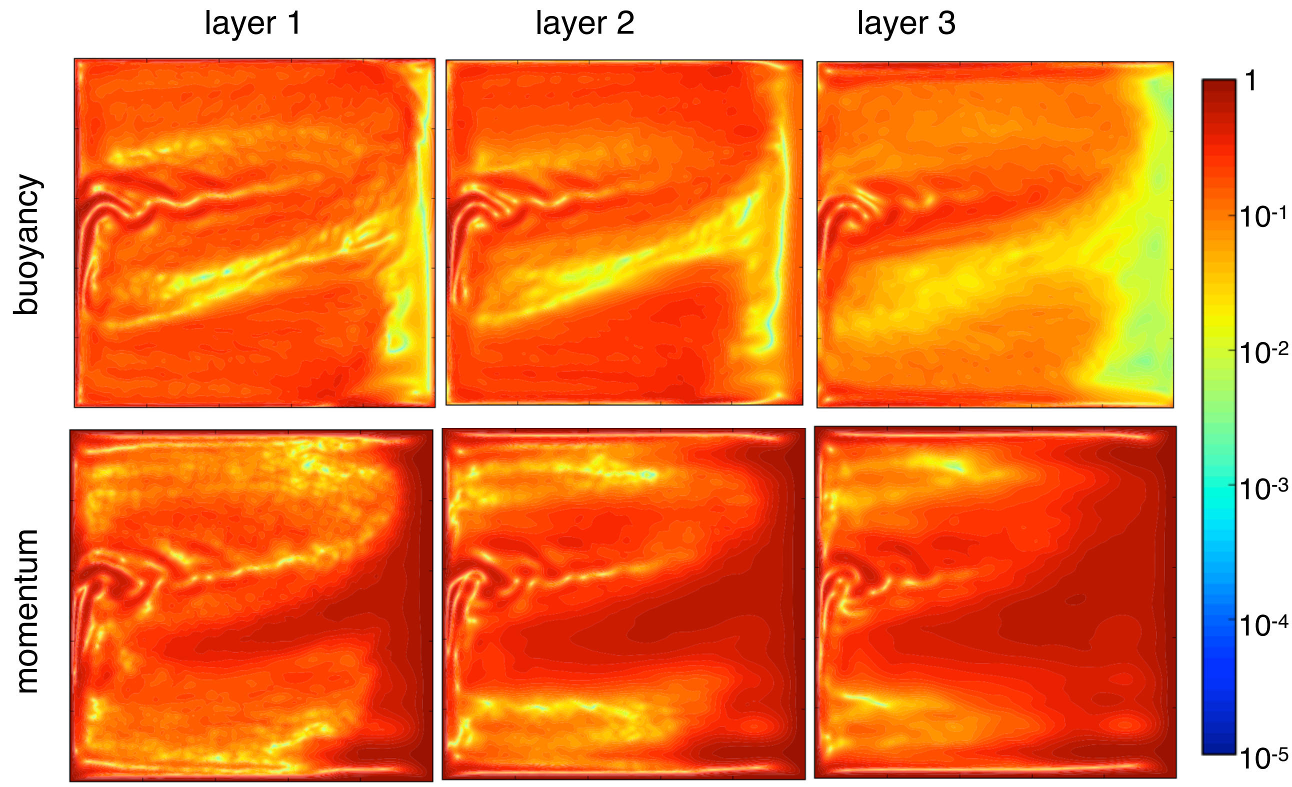
$-1 \le \alpha \le 1$ Eady growth rate if $\alpha = 0.61$

How anisotropic are the eddies?

3-layer QG model



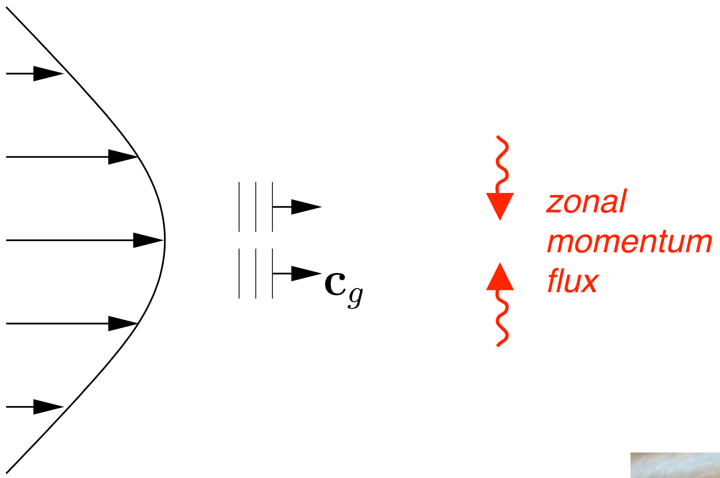
eddy anisotropies

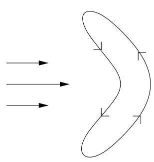


$$N = \gamma_m E \sin 2\phi_m \cos^2 \lambda \qquad S = \gamma_b \frac{f_0}{N_0} E \sin \phi_b \sin 2\lambda$$

What sets the eddy angles? - for linear Rossby waves: **refraction**

(e.g., Buhler and McIntryre, 2005)

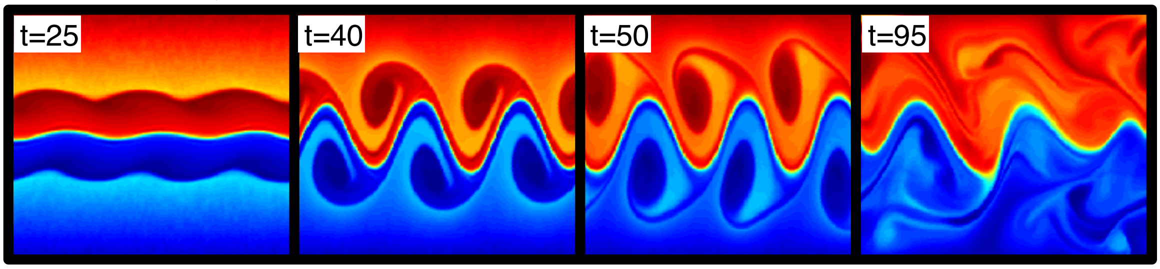


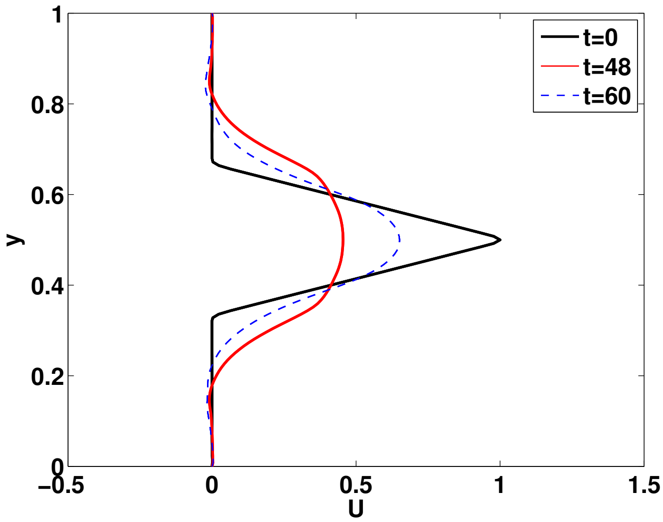


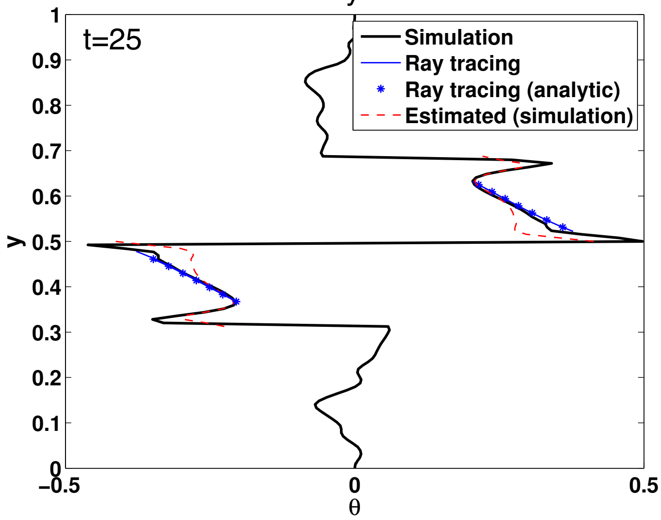
Ray-tracing - barotropic jet

Talia Tamarin

potential vorticity







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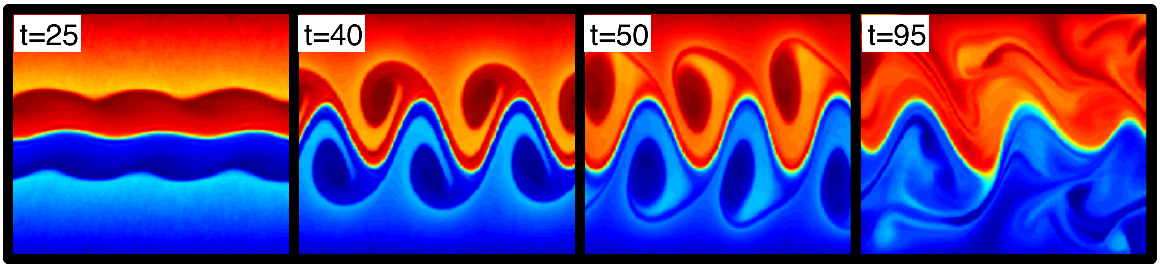
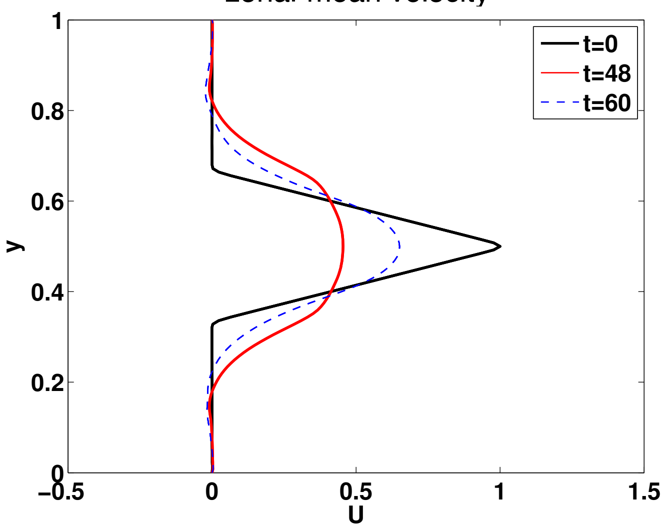
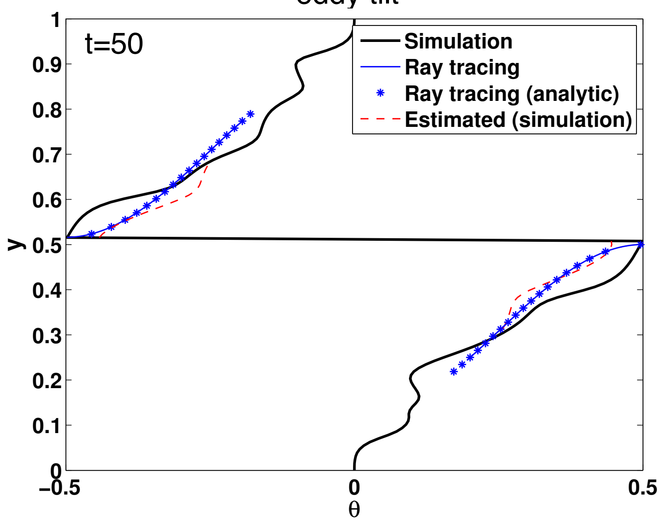






Mixing of potential vorticity?

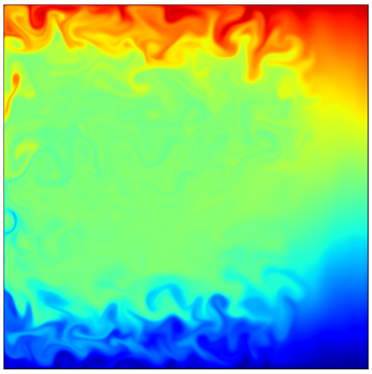
If we: (i) solve an eddy potential enstrophy ($\overline{q'^2}$) budget;

(ii) include dissipation of $\overline{q'^2}$ (= potential vorticity mixing);

(ii) ensure $\overline{q'\mathbf{u}'}$ vanishes when $\overline{q'^2}$ vanishes;

[use another bound on divergence of eddy stress tensor?]

then **Arnold's first stability theorem is preserved.**



Physical interpretation? (Marshall and Adcroft, 2010)

Eddy energy equation:

$$\frac{\partial}{\partial t}\frac{\overline{\mathbf{u}'\cdot\mathbf{u}'}}{2} + \nabla\cdot(\ldots) = +\overline{q'\mathbf{u}'}\cdot\nabla\overline{\psi}$$

Eddy enstrophy equation:

$$\frac{\partial}{\partial t}\frac{\overline{q'^2}}{2} + \nabla\cdot(\ldots) = -\overline{q'\mathbf{u}'}\cdot\nabla\overline{q}$$

If $d\overline{q}/d\overline{\psi} > 0$, eddy energy can grow only at the expense of eddy potential enstrophy.

$\Rightarrow$ stable (in the sense of Lyapunov) - **Arnold's first stability theorem**.

Coordinate-invariant derivation (Maddison and Marshall, 2013)

quasigeostrophic PV equation:

$$\partial_t\overline{q} + \left(\overline{[u_g]^a}\;\overline{q}\right)_{;a} = -T^{ab}_{;ab}$$

double divergence
$\Rightarrow$ 2 forms of gauge freedom

eddy flux tensor

$$g_{ac}T^{cb} = \begin{pmatrix} N & M-K & R \\ M+K & -N & S \\ 0 & 0 & 0 \end{pmatrix}$$

"residual-mean"

$$\begin{pmatrix} N & M+K & 0 \\ M-K & -N & 0 \\ R & S & 0 \end{pmatrix}$$

Cronin (1996)

$$\begin{pmatrix} N & M-K & 0 \\ M+K & -N & 0 \\ R & S & 0 \end{pmatrix}$$

"half-residual mean"

$$\begin{pmatrix} N & M & \frac{1}{2}R \\ M & -N & \frac{1}{2}S \\ \frac{1}{2}R & \frac{1}{2}S & 0 \end{pmatrix}$$

Plumb (1986)

$$\begin{pmatrix} N & M-P & 0 \\ M+P & -N & 0 \\ R & S & 0 \end{pmatrix}$$

Hoskins et al. (1983) "E-vector"

$$\begin{pmatrix} N & 2M & 0 \\ 0 & -N & 0 \\ R & S & 0 \end{pmatrix}$$

Approach generalises to isopycnal thickness-weighted primitive equations

(cf. Young, 2012)

Summary of key points

- Geostrophic eddies are fundamental in setting the structure and circulation of the ocean.

- Preserving symmetries and conservation laws in models with parameterised eddies
  $\Rightarrow$ classical stability conditions carry over.

- Down-gradient eddy potential vorticity flux closures are inconsistent with the underlying mathematical structure of the eddy-mean flow interaction.

- Gent and McWilliams is consistent with this underlying mathematical structure.

- New geometric framework for diagnosing and interpreting eddy-mean flow interactions.

- Much left to do, e.g.:

  - simple extension of Gent and McWilliams to include up-gradient momentum fluxes;
  - adjoint methods to optimise choice of parameters (Julian Mak);
  - eddy-topography interactions;
  - diagnostics of eddy-mean flow interactions in Southern Ocean (Andreas Klocker);
  - applications to planetary atmospheres?
  - ...

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