Dual constant-flux energy cascades in stably stratified rotating turbulence

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XSEDE grants ASC090050 & TG-PHY100029 and INCITE/DOE grant DE-AC05-00OR22725; * NSF/CMG 1025183

Where does energy go?

* T-HI (N=0, f=0): Direct or *(exclusive)* inverse energy cascade

* Examples of dual bi-directional constant-flux cascades

* Oceanic data and an apparent paradox

* Direct numerical simulations: process study for a range of parameters

* Conclusions and questions
- Direct numerical simulations: process study for a range of parameters

![Graph showing normalized dissipation rate \( D \) versus \( R_\lambda \). Direct numerical simulation data from Gotoh et al. (2002), Ishihara & Kaneda (2002), and Kaneda et al. (2003), together with those compiled by Sreenivasan (1998), i.e., the data from Cao et al. (1999), Jiménez et al. (1993), Wang et al. (1996), and Yeung & Zhou (1997). Figure redrawn from Kaneda et al. 2003.

The spectrum is of the form

\[
E(k) / \langle \varepsilon \rangle^{5/4} = \phi(k_\eta)
\]

in the wave-number range \( k \gg k_L \equiv 1/L \), and in particular

\[
E(k) \approx K_0 \langle \varepsilon \rangle^{2/3} k^{-5/3}
\]

in the inertial subrange \( k_L \ll k \ll k_d \), where \( \phi \) is a universal function of \( k_\eta \), \( k_d \equiv 1/\eta \), and \( K_0 \) is a nondimensional universal constant.

One can stringently examine Equation 2 by viewing a plot of the compensated spectrum \( \hat{E}(k_\eta) = k^{5/3} E(k) / \langle \varepsilon \rangle^{2/3} \) (Figure 3). If Equation 2 holds, the curves must be flat. The curves are nearly, but not strictly, flat at \( k_\eta \approx 0.01 \). The curves of \( \hat{E}(k_\eta) \) are close to each other at large \( k_\eta \) and \( R_\lambda \), in accordance with K41. The same is also true for the energy-flux \( \Pi_1(k) \) across wave number \( k \) defined as

\[
\Pi_1(k) = \int_0^\infty T(k) dk,
\]

where \( T(k) \) is the energy transfer function. A bump is observed in \( \hat{E}(k_\eta) \) at \( k_\eta \approx 0.1 \), but its height is lower for larger \( R_\lambda \). A similar, but less prominent, bump is also observed in the one-dimensional spectrum, \( E_1(k) \) (Gotoh et al. 2002, Saddoughi & Veeravalli 1994, Yeung & Zhou 1997).

The existence of a sufficiently wide inertial subrange \( k_L \ll k \ll k_d \) with \( \Pi_1(k) = \langle \varepsilon \rangle (3) \) is a prerequisite for theories and analyses of statistics in the inertial subrange. However, at \( R_\lambda \lesssim 200 \), such a range is not observed in Figure 3a. Misidentification of the range near the peak of the bump (i.e., \( k_\eta \approx 0.1 \)) as the inertial subrange results in an overestimate of the Kolmogorov constant \( K_0 \). \( k_\eta \) must be as small as \( \sim 0.01 \) to realize Equations 2 and 3. The plots give \( K_0 = 1.5–1.7 \). This value is close to the experimental value \( K_{\text{exp}} = 1.62 \) (Sreenivasan 1995) and consistent with DNSs by Gotoh & Fukayama (2001), Kaneda (2001), and Yeung & Zhou (1997). However,...

Boussinesq equations

\[ \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla P - N b e_z - 2\Omega e_z \times u + F \]
\[ \partial_t b + u \cdot \nabla b - \kappa \Delta b = N w , \]
\[ \nabla \cdot u = 0 . \]

Four dimensionless parameters:  \( \text{Re} = \frac{UL}{\nu} \gg 1 \)
\( \text{Pr} = \frac{\nu}{\kappa} = 1, \quad \text{Ro} = \frac{U}{[Lf]} \ll 1 \quad , \quad \text{Fr} = \frac{U}{[LN]} \ll 1 \)

\( R_B = \text{Re} \ \text{Fr}^2 \)

\( 2 \leq N/f \leq 10 \)

\( f = 2\Omega \)
Boussinesq equations

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Four dimensionless parameters:  
Re = UL/\nu \gg 1 
Pr = \nu/\kappa = 1,  
Ro = U/[Lf] \ll 1 ,  
Fr = U/[LN] \ll 1 

\[ R_B = Re \ Fr^2 \]
\[ 2 \leq N/f \leq 10 \]

f = 2\Omega
Spectra & fluxes for T-HI, of energy ____ & helicity ---

H= u.ω

Forced @ k=7

Navier Stokes, no rotation

Re=1200

1536^3 grid

Mininni & AP, 2009
Forced @ $k=100$

* Friction

Grid up to $16384^2$

$E(k) = C \varepsilon^{2/3} k^{-5/3}$, $C \approx 6$

$E(k) \sim k^{-(3+x)}$

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2D

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Boffetta 2007
We introduce a large-scale vorticity field cascades jointly and also the correlation between them. Following Chen, thanks to the resolution of the present simulations, we are able to analyse both cascade by using a filtering procedure recently introduced and applied separately to the direct be obtained by looking at the distribution of fluxes in space. This can be obtained vanishing of these definitions, balance equations for the large-scale energy.

Figure 1. Energy spectra for the two simulations for the direct cascade is simply proportional to injected is transferred to large scales where it is removed by friction at a rate again that the only e

Figure 2 shows the energy spectra computed for the different runs. We remark difficulties, there is a clear indication that the correction to the exponent is a finite-size e

Observe that because we change the resolution while keeping the ratio of increasing the resolution: this is a finite-size e

This fraction increases with the resolution and becomes about 95% for the run. The remaining energy injected is dissipated by viscosity at scales comparable with run. The remaining energy injected is dissipated by viscosity at scales comparable with

One of the simplest pieces of information which can be obtained from table 1 is that the extent of the inertial range is almost una

Unlike the inverse cascade, the direct enstrophy cascade is strongly a

Most of the enstrophy (around 90%) follows the direct cascade to small scales, while the extent of the inertial range regardless of the viscous dissipative scale, a

Robustness of the energy inertial range regardless of the viscous dissipative scale, a

vanishing of the energy with a Kolmogorov constant

$E(k) \sim k^{-3+\delta}$

$E(k) \sim k^{-5/3}$

forced @ $k=100$

Grid up to $16384^2$

Boffetta 2007
Paradigm with 2 invariants like energy & enstrophy:

2D: Dual but mutually exclusive system with an inverse cascade of energy & a direct enstrophy cascade

3D: Direct cascade of energy, and direct helicity cascade

BUT …
3D, T-HI, 2D2C force, $A = L_z/L_x = 1/64$ with $S = L_f/L_z$

Turbulent viscosity, Navier-Stokes, no rotation, $128^3$ grid

FIG. 2. (upper) $A = 1/64$, $Ro = \infty$, $S = 0.75$ (statistically steady); (lower) $A = 1/64$, $Ro = \infty$, $S = 0.375$: eddy viscosity (solid line) with time increasing upwards; hyperviscosity (dotted line). The lines are $E_h \propto k_h^{-5/3}$.

Smith et al. PRL 1996
(also: Celani et al. PRL 2010)
- Physical systems with dual cascades
Acoustic turbulence in superfluid He

Energy in low frequency: evidence for an inverse cascade

Fig. 3. (A) Transient evolution of the second sound wave amplitude $\delta T$ after a step-like shift of the driving frequency to the 96th resonance at time $t = 0.397$ s. Formation of isolated “rogue” waves is clearly evident. (Inset) Example of a rogue wave, enlarged from frame 2. (B) Instantaneous spectra in frames 1 and 3 of A. The lower (blue) spectrum, for frame 1, shows the direct cascade only; the upper (orange) spectrum, for frame 3, shows both the direct and inverse cascades. The green arrow indicates the fundamental peak at the driving frequency. (Inset) Evolution of the wave energy in the low-frequency and high-frequency domains is shown by the orange squares and blue triangles respectively; black arrows mark the positions of frames 1 and 3. (After ref. 72.)

Kolmakov et al. 2014,
After Ganshin et al. 2008
Kinetic energy flux in 3D MHD for various $V_A$

![Graph showing kinetic energy flux in 3D MHD for various $V_A$.](Image)

**Legend:**
- R1 ($V_A = 5$)
- R2 ($V_A = 2$)
- R3 ($V_A = 10$)

**Graph Details:**
- The graph plots $\Pi_\perp/\epsilon_{\text{inj}}$ against $k_\perp$.
- $\Pi_\perp$ is the energy flux perpendicular to the magnetic field.
- $\epsilon_{\text{inj}}$ is the injection rate of energy.
- $k_\perp$ is the wavenumber perpendicular to the mean magnetic field.

**Observations:**
- As $V_A$ increases, the energy flux $\Pi_\perp$ becomes more negative, indicating a stronger inverse cascade.
- The direct cascade is weaker as $V_A$ increases.

**Discussion:**
- With increased $V_A$, the flow becomes closer to a two-dimensional flow, as evidenced by the energy flux and energy spectrum behavior.
- The kinetic energy in the large scales of kinetic energy is significantly smaller than that of the magnetic energy, as expected.
- The spectra for these runs are compared in Fig. 4, which shows the energy flux in the perpendicular direction.

**References:**
- Alexakis, 2011

**Keywords:**
- Kinetic energy flux
- 3D MHD
- $V_A$
- Energy spectra
Energy flux in anisotropic shell models

FIG. 2. Energy flux $\Pi(k_n)$ normalized with the input $\epsilon_I$ for increasing values of $k_h/k_f$ from top to bottom: $k_h/k_f = 2^0, 2^1, 2^3, 2^5, 2^{15}$. The shell $k_h$ is indicated by black dots on each curve. Inset: Energy flux in the direct energy cascade $\epsilon_v$ as a function of the scale separation $k_h/k_f$. Dashed line represents the prediction $\epsilon_v/\epsilon_I \sim (k_h/k_f)^{-\beta}$.

Boffetta et al. 2011
What happens with rotation and stratification in an idealized setting?
Figure 3-1: Buoyancy frequency ($s^{-1}$) in logarithmic scale from the ALBATROSS section, Drake Passage.

Figure 3-2: Flow speed ($m\ s^{-1}$) from the ALBATROSS section, Drake Passage.

Measurements in the Southern Ocean

← of flow speed

and of buoyancy frequency

-Nikurashin, 2009-
The Southern Ocean is driven by surface fluxes of momentum and density (owing to heat and fresh water) induced by the strong, predominantly westerly winds that blow over it and the freezing phenomena close to the continent. Zonal wind stress (Fig. 4a) induces upwelling polewards of the zonal surface-wind maximum and downwelling equatorwards of the maximum. This directly wind-driven circulation, known as the Deacon cell, acts to overturn density surfaces supporting the thermal wind current of the ACC and creating a store of available potential energy. Air–sea fluxes generate dense water near the continent and lighten the surface layers in the ACC (see Fig. 4b). The dense water sinks and tends to draw in warmer, saltier water from the surrounding ocean; however, rather than being fed from the surface, it is fed from below through the dense water formation.
Kinetic energy flux in the ACC, 10+yrs data every 10 days \( \sim T_{NL} \),

Deformation radius

Scott 2005
Energy flux and spectrum

ROMS
Forcing in momentum, fresh water & heat with restoring force, KPP & sponge layer
Down to 0.75km res. (solid line)

Larger range for the inverse cascade than for the direct one

Capet et al., 2008
5. Spectral flux of kinetic energy ($Re_{eff} = 6600$): BOUS ($Ro_r = 0.5$, black) and QG (blue). Note the forward cascade for BOUS and inverse cascade for QG.

Molemaker et al. 2010
A paradox?

- **Capet et al. (2008), ROMS+KPP:**
  
  ... we hesitate to draw any strong conclusions about the efficacy of a mesoscale inverse KE \(\{\text{Kinetic Energy}\}\) cascade in our solutions, although our results indicate it does occur to some degree ...

* **Scott et al. (2011), oceanic data analysis:**
  
  ... despite great effort in studying the ocean’s energy budget in the last two decades, the bulk of the dissipation of the most energetic oceanic motions remains unaccounted for.

* **Arbic et al. (2013), oceanic data and modeling:**
  
  ... It is therefore difficult to say whether the forward cascades seen in present-generation altimeter data are due to real physics (represented here by eddy viscosity) or to insufficient horizontal resolution.
Geophysical High Order Suite for Turbulence (Gomez & Mininni)

- Pseudo-spectral DNS, periodic BC cubic (also 2D), single/double precision; Runge-Kutta for incompressible Navier-Stokes, SQG & Boussinesq. Includes rotation, passive scalar(s), MHD + Hall term
- GHOST, from laptop to high-performance, parallelizes linearly up to 100,000 processors, using hybrid MPI/Open-MP (Mininni et al. 2011, Parallel Comp. 37)
- 3D Visualization: VAPOR (NCAR); and development @ OakRidge (D. Rosenberg)
- LES: alpha model & variants (Clark, Leray) for fluids & MHD
- Helical spectral (EDQNM) model for eddy viscosity & eddy noise
- NEW! Lagrangian particles (w. A. Pumir, ENS)
- NEW! Gross-Pitaevskii & Ginzburg-Landau (with M. Brachet, ENS)
- Data, forced: 2048³ Navier-Stokes and 1536³ & 3072³ with rotation, both w. or w/o helicity. Rotating stratified turbulence w. 2048³ grids.
- Spin-down MHD: 1536³ random + 6144³ ideal & 2048³ w. T-Green symmetry.
- Decaying rotating stratified flow, N/f~5, Re=5.5 10⁴, 2048³, 3072³ & 4096³ grids

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Rotating-stratified data

Runs:
Raffaele Marino (NCAR) & Duane Rosenberg (OakRidge)

Diagram:
Corentin Herbert (Weizmann)

CPU: NSF (XSEDE & Yellowstone/NCAR); *: DOE/INCITE, 4096³ grid

Forcing at $K_F \sim 10$

Small-scale spectra
$N/f =2$ and for different parameters

$R_B = Re \cdot Fr^2$

$R_B = 6, 40 \text{ & } 80$

$R_B = 120, E(k) \sim k^{-1.77}$
Large-scale spectra, $N/f=2$

Temporal growth of energy (_____) & stabilisation of energy dissipation (- - - - )
recover isotropy for high enough latitudes has 
However, we note that the abyssal southern ocean at mid

when examining AVISO altimeter data for the Kuroshio
turbulence have been analyzed using theoretical closure
occurs in the purely stratified case

Reynolds number is indicative of the increased effective-
the ratio of inverse to direct flux with the buoyancy
transient, is typical of inverse cascades. The variation of
(a) Total (kinetic plus potential) energy fluxes normalized by energy input
(b) Energy spectra for Run 10d (red line), 10e (blue line), and 15a (black line), all with
PRL 5 (2013)

N/f=2
N/f=4

Pouquet & Marino,
PRL 2013
Grids of $1024^3$, $1536^3$ & $2048^3$ points, $K_F=\{10,11\}$

<table>
<thead>
<tr>
<th>Run</th>
<th>$Re$</th>
<th>$Fr$</th>
<th>$Ro$</th>
<th>$N/f$</th>
<th>$R_B$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10a</td>
<td>5000</td>
<td>0.020</td>
<td>0.08</td>
<td>4</td>
<td>2.0</td>
<td>-3.99</td>
</tr>
<tr>
<td>10b</td>
<td>5000</td>
<td>0.045</td>
<td>0.18</td>
<td>4</td>
<td>10.1</td>
<td>-2.93</td>
</tr>
<tr>
<td>10c</td>
<td>5000</td>
<td>0.060</td>
<td>0.24</td>
<td>4</td>
<td>18.0</td>
<td>-2.34</td>
</tr>
<tr>
<td>10d</td>
<td>4000</td>
<td>0.040</td>
<td>0.08</td>
<td>2</td>
<td>6.4</td>
<td>-3.99</td>
</tr>
<tr>
<td>10e</td>
<td>5000</td>
<td>0.090</td>
<td>0.18</td>
<td>2</td>
<td>40.5</td>
<td>-2.12</td>
</tr>
<tr>
<td>15a</td>
<td>8000</td>
<td>0.100</td>
<td>0.20</td>
<td>2</td>
<td>80.0</td>
<td>-1.87</td>
</tr>
</tbody>
</table>

*20a  12000  0.1  0.2  2  120  1.05  -1.77

$Re=UL/\nu$, $Fr=U/[LN]$, $Ro=U/[Lf]$

$R_B=ReFr^2$

$R_\tau = \varepsilon_l/\varepsilon_D$

$E(k) \sim k^{-\alpha}$

Pouquet & Marino, PRL 2013
Horizont. slice of vertical velocity

R=8000
N/f=2
Fr=0.1
R_B=80
1536^3 res.

Forcing scale

Pouquet & Marino PRL 2013


TWO different compensations for the kinetic energy spectrum

\[ N/f=7, \; Fr=0.047 \]

\[ k_F = 10, 11 \]

\[ E_V(k) / \alpha \varepsilon_V^{2/3} k^{\sigma} \]

\[ t / \tau_{NL} \sim 29 \]
Kolmogorov constant for the inverse cascade

\[ E_v(k) / \varepsilon^{2/3} k^{-5/3} \]

\[ K_{\text{forcing}} = 10, 11 \]
$\Pi T/\varepsilon_V^{0.6}$

Ro=0.26, variable Fr
The small-scale flux is smaller when waves are stronger.

\[ \text{Regime of ``weak'' turbulence} \]

\[ \varepsilon_{\text{dir,WT}} = \varepsilon_{\text{Kol}} \times Fr \]
The **small-scale** flux is smaller when waves are stronger

→ Regime of ``weak'' turbulence

→ \( \varepsilon_{\text{dir,WT}} = \varepsilon_{\text{Kol}} \times Fr \)

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Energy flux

<table>
<thead>
<tr>
<th>N/f=2</th>
<th>N/f=4</th>
</tr>
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</table>

- \( \text{Re}=4,000 \)  \( \text{Fr}=0.04 \)  \( \text{Ro}=0.08 \)
- \( \text{Re}=5,000 \)  \( \text{Fr}=0.09 \)  \( \text{Ro}=0.18 \)
- \( \text{Re}=8,000 \)  \( \text{Fr}=0.1 \)  \( \text{Ro}=0.2 \)
- \( \text{Re}=5,000 \)  \( \text{Fr}=0.02 \)  \( \text{Ro}=0.08 \)
- \( \text{Re}=5,000 \)  \( \text{Fr}=0.045 \)  \( \text{Ro}=0.18 \)
- \( \text{Re}=5,000 \)  \( \text{Fr}=0.06 \)  \( \text{Ro}=0.24 \)
Fr = 0.04, variable Ro
N/f = Ro/Fr, Ro = U/[Lf]
Ro ~ 0.08
~ 0.16
~ 0.28
~ 0.45

The stronger the rotation, the larger is $R_\Pi$, i.e. the larger is the cascade to large scales relative to that to small scales.
\[ \varepsilon_{LS} / \varepsilon_{SS} \sim [Fr \times Ro]^{-1} \]

\[ \sim \omega_{rms} [Nf]^{-1/2} Re^{-1} \]
\[ R_\Pi = \frac{\varepsilon_{\text{inv}}}{\varepsilon_{\text{dir}}} \]

* Point labeled with values of \( R_B = \text{Re} \, \text{Fr}^2 \)
\[ R_{\Pi} = \frac{\varepsilon_{\text{inv}}}{\varepsilon_{\text{dir}}} \]

\[ R_B = \text{Re Fr}^2 \]
Is $N/f \sim 7$ special? Why a transition there?
Inverse cascade of potential energy?

\[ E_P(k) / \alpha k^\sigma \]

\[ k_F = 10,11 \]

\[ t / \tau_{NL} \sim 29 \]

\[ \Pi / \varepsilon_v \]

\[ N/f = 2 \quad R_{\Pi} = 22.4 \]
\[ N/f = 4 \quad R_{\Pi} = 4.4 \]
\[ N/f = 7 \quad R_{\Pi} = 3.5 \]
\[ N/f = 10.5 \quad R_{\Pi} = 1.7 \]
Is there an enstrophy cascade?

\[ \frac{\Pi}{\epsilon_V} \quad (5 \cdot 10^{-2}) \times \frac{\Pi_{\text{lof}}^2}{\epsilon_V k_F^2} \]
Summary, future work and open questions

Scaling with \([\text{Fr}^*\text{Ro}]^{-1}\) of the flux ratio of the bi-directional cascade

- Anisotropic analysis & normal modes decomposition
- Role of helicity? Role of conservation of potential vorticity?
- Cascade of enstrophy? Of potential energy?

- Long-time accumulation at \(k=1\), & large-scale friction?
- Different forcing, e.g. two-dimensional or balanced?
- Criticality?

- Lagrangian particles, mixing and passive scalar in a dual cascade

Different regimes: What are the characteristic break-up scales for energy partition, and how do such scales vary with parameters?

- Modeling with anisotropic eddy viscosity (>0, <0)?
Thank you for your attention