# Statistical mechanics and the vertical structure of geostrophic turbulence

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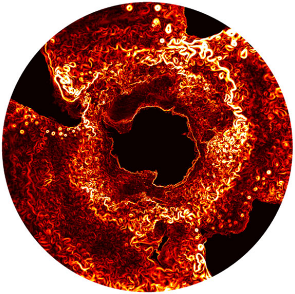
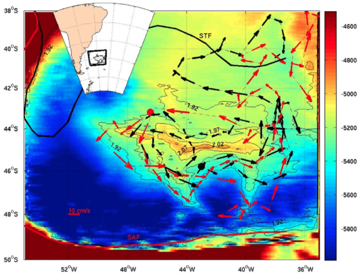
# Self-organization at mesoscale in the oceans



*Surface Kinetic Energy, from ECCO2*




*Zapiola Anticyclone, 100 Sv from Saraceno et al 2009*

**What sets the horizontal and vertical shape of these flows ?**

**Here: a statistical mechanics approach in idealized settings**

# Outline

1. **Vertical partition of the energy in stratified QG turbulence**
   -How to transfer the energy of surface intensified eddies into bottom trapped recirculations above a topographic bump?

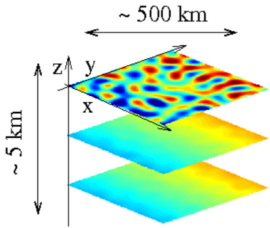
2. **Bottom friction**
   -Can we understand the condensation of kinetic energy into surface intensified "ribbons" in a large bottom friction limit?

# Stratified QG flows

$$\partial_t q_i + \mathbf{u}_i \cdot \nabla q_i = 0, \quad \mathbf{u}_i = (-\partial_y \psi_i,\ \partial_x \psi_i)$$

Coupling parameter $F = \left(\frac{f}{N\delta h}\right)^2$:

$$\begin{aligned} q_1 &= \nabla^2\psi_1 + F\left(\psi_2 - \psi_1\right) + \beta y \\ q_i &= \nabla^2\psi_i + F\left(\psi_{i+1} + \psi_{i-1} - 2\psi_i\right) + \beta y \\ q_n &= \nabla^2\psi_n + F\left(\psi_{n-1} - \psi_n\right) + \frac{f}{\delta h} h_b(\mathbf{r}) + \beta y \end{aligned}$$




**For an initial surface intensified unstable velocity field, what is the effect of $\beta y$ and $h_b$ on the final state organization ?**

# Dynamical invariants

$$\partial_t q_i + \mathbf{u}_i \cdot \nabla q_i = 0, \quad \mathbf{u}_i = (-\partial_y \psi_i,\ \partial_x \psi_i)$$

**Energy** (global)

$$\mathcal{E}[q] = \frac{\delta h}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r} \ \left( \sum_{i=1}^{n} (\nabla \psi_i)^2 + \sum_{i=1}^{n-1} F \left(\psi_i - \psi_{i+1}\right)^2 \right)$$

**Potential vorticity** (layerwise)

$$\mathcal{C}_s[q_i] = \int_{\mathcal{D}} \mathrm{d}\mathbf{r} \ s\left(q_i\right)$$

**Including potential enstrophy**

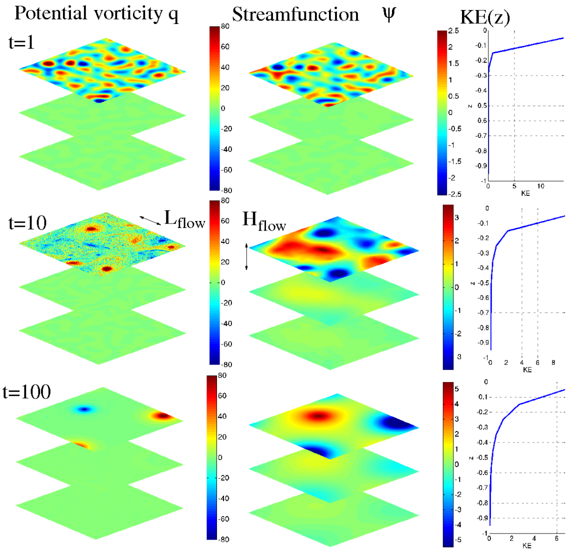
$$\mathcal{Z}[q_i] = \frac{1}{2} \int_{\mathcal{D}} \mathrm{d}\mathbf{r} \ q_i^2$$

# Surface quasi-geostrophic flows

$$\partial_t q_i + \mathbf{u}_i \cdot \nabla q_i = 0, \quad \mathbf{u}_i = (-\partial_y \psi_i,\ \partial_x \psi_i)$$

$$\begin{aligned} q_1 &= \nabla^2 \psi_1 + F\left(\psi_2 - \psi_1\right) \\ 0 &= \nabla^2 \psi_i + F\left(\psi_{i+1} + \psi_{i-1} - 2\psi_i\right) \\ 0 &= \nabla^2 \psi_n + F\left(\psi_{n-1} - \psi_n\right) \end{aligned}$$

# Numerical simulation




**The energy remains surface intensified**

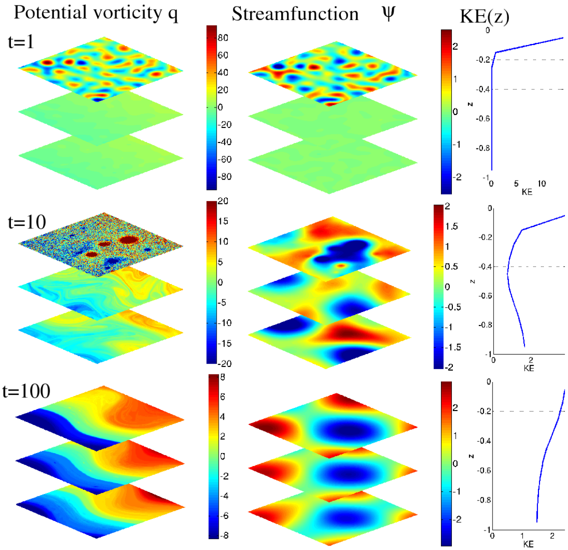
$$H_{flow} = \frac{f}{N} L_{flow}$$

# Switch on beta

$$\partial_t q_i + \mathbf{u}_i \cdot \nabla q_i = 0, \quad \mathbf{u}_i = (-\partial_y \psi_i,\ \partial_x \psi_i)$$

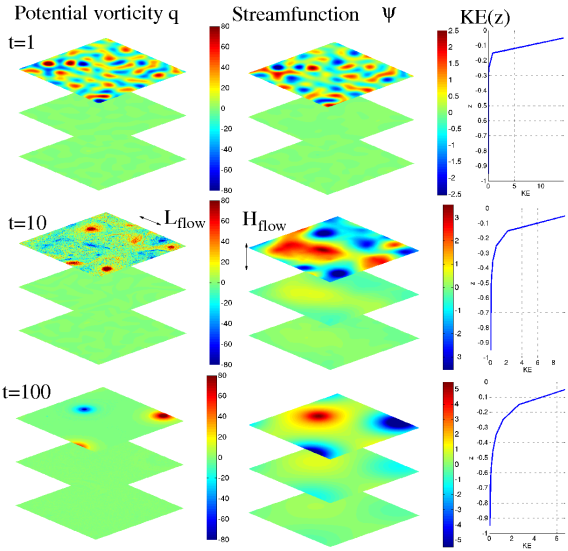
$$\begin{aligned}
q_1 &= \nabla^2 \psi_1 + F(\psi_2 - \psi_1) + \beta y \\
q_i &= \nabla^2 \psi_i + F(\psi_{i+1} + \psi_{i-1} - 2\psi_i) + \beta y \\
q_n &= \nabla^2 \psi_n + F(\psi_{n-1} - \psi_n) + \beta y
\end{aligned}$$

# With beta effect




**This allows to spread the energy on the vertical.**

# Without beta effect



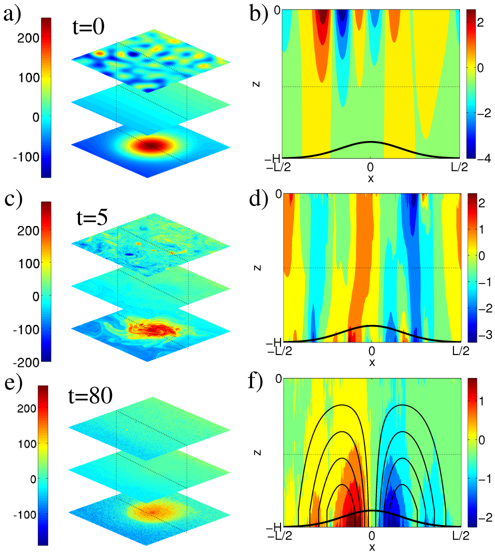

**The energy remains surface intensified**

# Switch on bottom topography

$$\partial_t q_i + \mathbf{u}_i \cdot \nabla q_i = 0, \quad \mathbf{u}_i = (-\partial_y \psi_i,\ \partial_x \psi_i)$$

$$\begin{aligned} q_1 &= \nabla^2 \psi_1 + F(\psi_2 - \psi_1) \\ 0 &= \nabla^2 \psi_i + F(\psi_{i+1} + \psi_{i-1} - 2\psi_i) \\ q_n &= \nabla^2 \psi_n + F(\psi_{n-1} - \psi_n) + \frac{f}{\delta h} h_b(\mathbf{r}) \end{aligned}$$

# With topography




**Formation of bottom trapped flows along isobath.**

# Summary

$$\partial_t q_i + \mathbf{u}_i \cdot \nabla q_i = 0, \quad \mathbf{u}_i = (-\partial_y \psi_i,\ \partial_x \psi_i)$$

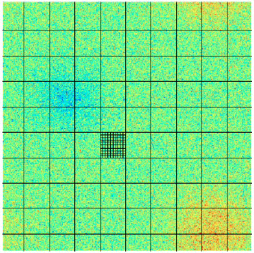
$$\begin{aligned} q_1 &= \nabla^2 \psi_1 + F(\psi_2 - \psi_1) + \beta y \\ q_i &= \nabla^2 \psi_i + F(\psi_{i+1} + \psi_{i-1} - 2\psi_i) + \beta y \\ q_n &= \nabla^2 \psi_n + F(\psi_{n-1} - \psi_n) + \frac{f}{\delta h} h_b(\mathbf{r}) + \beta y \end{aligned}$$

**Planetary vorticity gradients $\beta y$ and bottom topography $h_b$ provide available potential vorticity levels that may be stirred, which in turn modifies the flow dynamics. It allows for energy transfers from the surface to the interior**

# Equilibrium statistical mechanics of QG flows

**The observed large scale flow is the most probable state among all the configurations satisfying the constraints of the problem.**

An overwhelming number of *microscopic configurations* $q_i$ correspond to the most probable coarse-grained vorticity field $\overline{q}_i$



Point vortex models: *Onsager 49*
Truncated models: *Kraichnan 67 Salmon Holloway Hendershott 76*
Continuous Euler and QG dynamics *Miller 91 Robert Sommeria 91*,...
Stratified QG *Merryfield 98, Schecter 05*

The theory predicts a relation $\overline{q}_i = g_i(\psi_i)$.

# Strong mixing limit for stratified QG

- Equilibria are solutions of

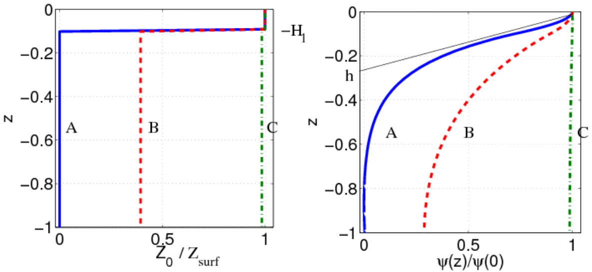
$$\min_{\{\overline{q}_i\}} \left\{ \sum_{i=1}^{n} \frac{\frac{1}{2}\int \mathrm{d}x\mathrm{d}y\ \overline{q}_i^2}{Z_{i0}} \mid \mathcal{E}\left[\overline{q}\right] = E_0 \right\}$$

- Those states are characterized by $\overline{q}_i = \lambda Z_{i0} \psi_i$.

*Venaille Vallis Griffies JFM 12*. See *Herbert PRE 14* in the two layer case.

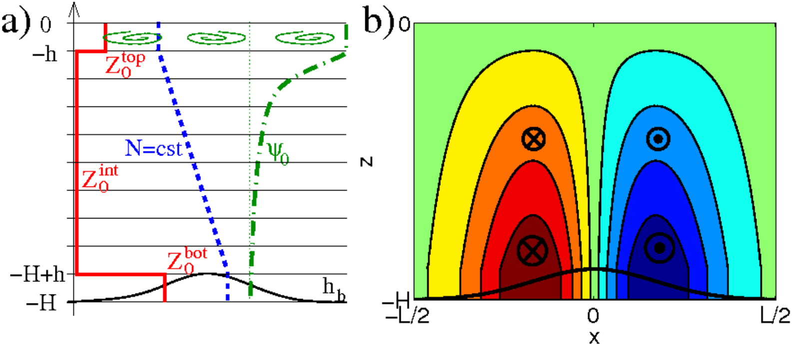
# Role of beta effect

$$\overline{q} = \nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial}{\partial z}\psi\right) + \beta_c y, \quad \overline{q} = \lambda Z_0(z)\psi$$




$Z_0$ **depth independant** $\Rightarrow$ **depth independant equilibrium state.**

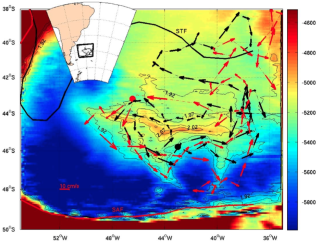
# Role of bottom topography




$Z_0$ **bottom intensified: bottom intensified equilibrium state.**

*Venaille JFM 12*, see also *Dewar JMR 98, Merryfield JFM 98*.

# A route for energy dissipation




**Wind**

→ surface intensified mean flow

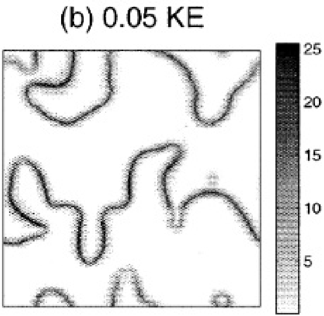
→ baroclinic instability

→ geostrophic turbulence

→ bottom-intensified mean flow

**Bottom friction**

# Bottom friction

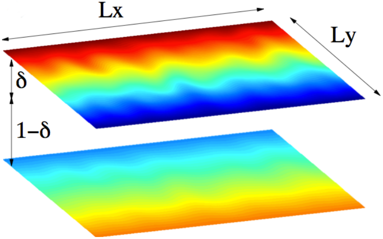



**Ribbons are sharp, meandering and surface intensified jets.** They appear in a large bottom friction limit.

*From Arbic & Flierl, JPO 04*
*Thompson Young JPO 06*

# 2 layers QG flow in a channel

$$\partial_t q_1 + J(\Psi_1, q_1) = 0,$$
$$\partial_t q_2 + J(\Psi_2, q_2) = -r\nabla^2\Psi_2$$




$$q_1 = \nabla^2\Psi_1 + \frac{\Psi_2 - \Psi_1}{\delta R^2},$$
$$q_2 = \nabla^2\Psi_2 + \frac{\Psi_1 - \Psi_2}{(1-\delta) R^2},$$

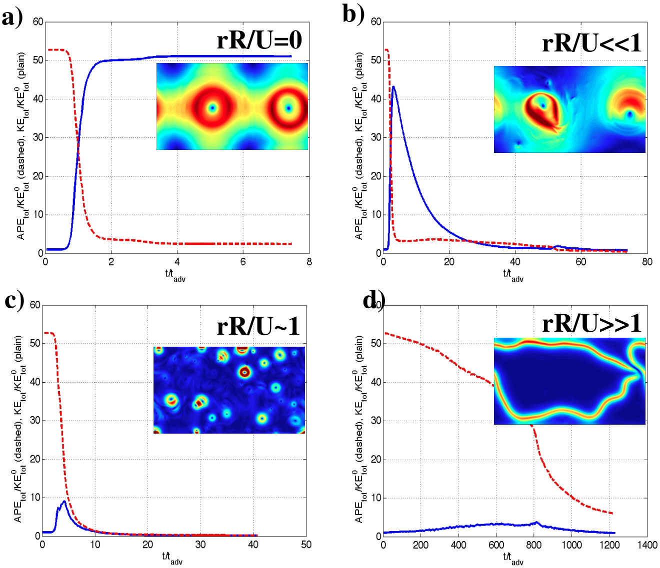
**Initial condition**: eastward jet in the upper layer
$\Psi_1 = -Uy,\ \Psi_2 = 0.$

**Energy budget** $\mathrm{d}_t E = -r \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y\ (1-\delta)(\nabla\Psi_2)^2.$

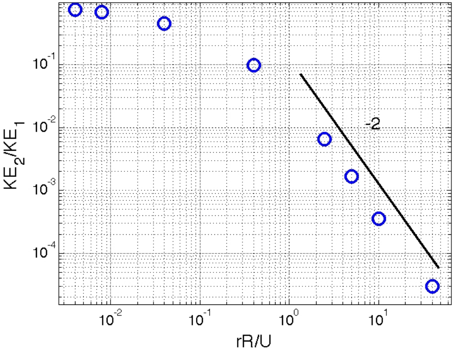
# Parameters

1. **Bottom friction $rR/U$**
2. Rossby radius $R/L_y \ll 1$
3. Vertical aspect ratio $\delta$
4. Horizontal aspect ratio $L_x/L_y \sim 1$

# Temporal evolution of the energy

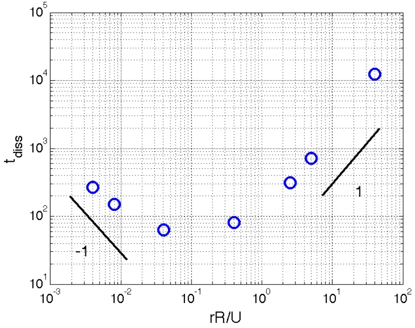


# Vertical flow structure




- **Barotropization** for weak frictions: $\Psi_1 = \Psi_2$
- $1^1/_2$ **QG dynamics** for large frictions: $\Psi_1 \gg \Psi_2$.

# Time scale for energy dissipation




- For weak friction, barotropization implies $\partial_t E = r(1-\delta)E$
- **For large friction, increasing bottom friction decreases the time scale for energy dissipation**

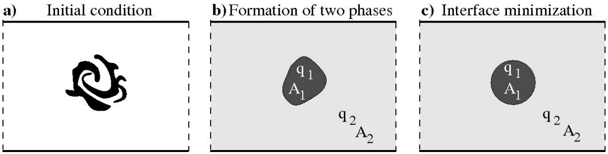
# Ribbons in 1½ QG turbulence

$$\partial_t q_1 + J(\Psi_1, q_1) = 0, \quad q_1 = \nabla^2 \Psi_1 - \frac{\Psi_1}{\delta R^2}$$

**Cascade argument for condensation of the kinetic energy into ribbons of width** $R$ *Venaille Nadeau Vallis 2014*

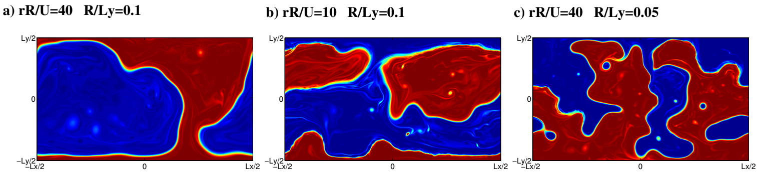
**Equilibrium states in the small** $R$ **limit** *Bouchet Sommeria 2002*

- Formation of different phases of homogenized PV with strong jets of width $R$ at their interface.
- Interfaces "cost" free energy



# The interface length

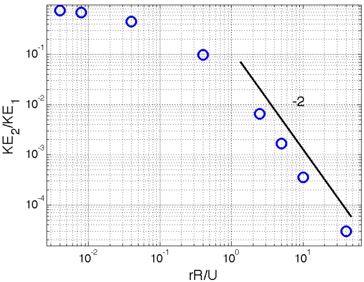
- $1^1/2$ QG dynamics: tendency to reach a state with a minimal interface
- Baroclinic instability of the jet: destabilization of the interface

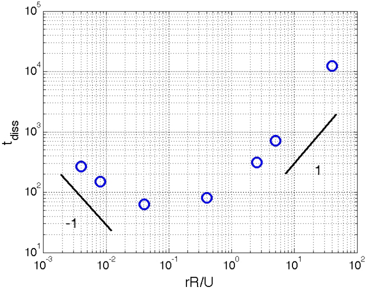



$$L_{blob} \sim L_y \left(\frac{rR}{U}\right)^{1/4} \left(\frac{R}{L_y}\right)^{1/2}$$

# Time scale for energy dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{eddy} = \frac{U}{R^2} \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y \; \psi_1 \partial_x \psi_2 - r \int_{\mathcal{D}} \mathrm{d}x\mathrm{d}y \; (1-\delta) \left(\nabla \psi_2\right)^2$$



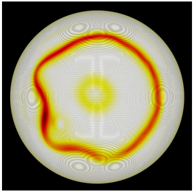




$$\frac{\psi_2}{\psi_1} \sim \left(\frac{rR}{U}\right)^{-1}, \qquad t_{diss} \sim r \left(\frac{R}{U}\right)^2 \frac{L_y}{R}$$

# Prospects: forced-dissipated case

**Relaxation toward radiative equilibrium on a sphere** *(with B. Marston)*

$$\begin{aligned}\partial_t q_1 + J(\Psi_1, q_1) &= -\alpha(\psi_c - \psi_c^*), \\ \partial_t q_2 + J(\Psi_2, q_2) &= \alpha(\psi_c - \psi_c^*) - r\nabla^2\Psi_2\end{aligned}$$



# Conclusion

1. **Planetary vorticity gradients** favor barotropization.
   *Venaille Vallis Griffies JFM 2012*
2. Energy transfer from surface-intensified eddies to bottom trapped recirculations above **topographic anomalies**
   *Venaille JFM 2012*
3. In the limit of **very large bottom friction**, kinetic energy is condensed into **surface intensified, meandering jets** of width given by the baroclinic Rossby radius of deformation.
   *Venaille Nadeau Vallis, Physics of Fluids 2014*
4. Those dynamical features may be understood at a qualitative level with **statistical mechanics**.
   *Bouchet Venaille, Physis Reports 2012*
5. Ongoing studies in forced-dissipated configurations.

# What about the ocean ?

- Exemple from Nadeau Straub JPO 2009
  $r \sim 10^7 \text{ s}^{-1}$, $R \sim 50$ km, $U_{sverdrup} \sim 0.01 \text{ m.s}^{-1}$
- Arbic and Flierl JPO 2004 estimate $rR/U \sim 1$ in the oceans.
- Lacasce Brink JPO 1999 observed that increasing **bottom topography** sometimes (indirectly) amounts to increasing bottom friction (so that $r_{effective}R/U \gg rR/U$)
- Stratification, small scale topography,...

# Cascade phenomenology

$$\partial_t q_1 + J(\Psi_1, q_1) = 0, \quad q_1 = \nabla^2 \Psi_1 - \frac{\Psi_1}{\delta R^2}$$

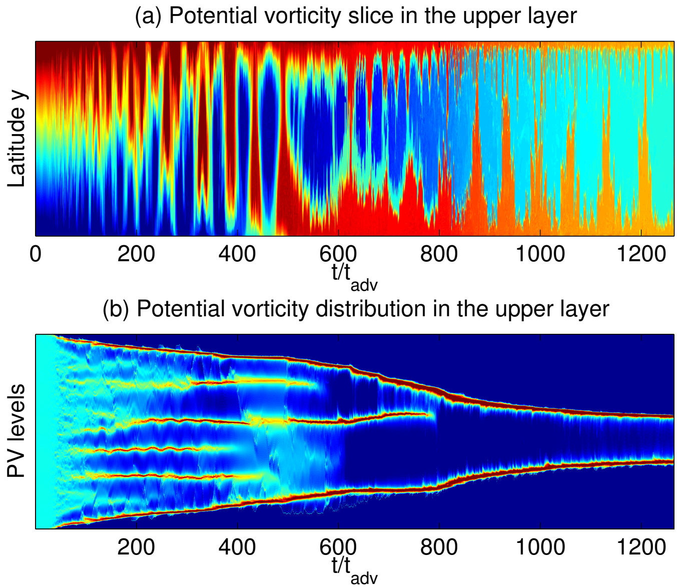
**Small scales** $q_1 \approx \nabla^2 \Psi_1$,

$KE_1 = -\int \mathrm{d}\mathbf{r}\ \Psi_1 \nabla^2 \Psi_1$ cascades towards **large** scales.
$Z_1 = \int \mathrm{d}\mathbf{r}\ \left(\nabla^2 \Psi_1\right)^2$ cascades towards **small** scales.

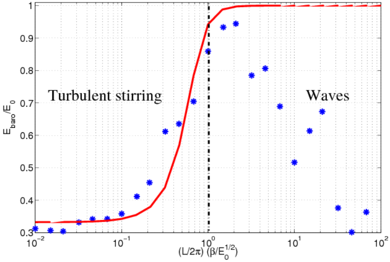
**Large scales** $q_1 \approx -\Psi_1/\delta R^2$,

$KE_1 = -\int \mathrm{d}\mathbf{r}\ \Psi_1 \nabla^2 \Psi_1$ cascades towards **small** scales.
$APE = \int \mathrm{d}\mathbf{r}\ \Psi_1^2$ cascades towards **large** scales.

# Emergence of the ribbons



# The catalytic role of beta effect in barotropization process




- *Charney 71* "vertical inverse cascade" in a case with constant enstrophy on the vertical.
- *Rhines 77* coined the term barotropization.
- Numerical studies by Hua Haidvogel JPO 88, McWilliams Science 94, *Smith and Vallis JPO 01*,...